2.3 | The Limit Laws

Learning Objectives

2.3.1 Recognize the basic limit laws.
2.3.2 Use the limit laws to evaluate the limit of a function.
2.3.3 Evaluate the limit of a function by factoring.
2.3.4 Use the limit laws to evaluate the limit of a polynomial or rational function.
2.3.5 Evaluate the limit of a function by factoring or by using conjugates.
2.3.6 Evaluate the limit of a function by using the squeeze theorem.

In the previous section, we evaluated limits by looking at graphs or by constructing a table of values. In this section, we establish laws for calculating limits and learn how to apply these laws. In the Student Project at the end of this section, you have the opportunity to apply these limit laws to derive the formula for the area of a circle by adapting a method devised by the Greek mathematician Archimedes. We begin by restating two useful limit results from the previous section. These two results, together with the limit laws, serve as a foundation for calculating many limits.

Evaluating Limits with the Limit Laws

The first two limit laws were stated in **Two Important Limits** and we repeat them here. These basic results, together with the other limit laws, allow us to evaluate limits of many algebraic functions.

Theorem 2.4: Basic Limit Results

For any real number a and any constant c,

i. $$\lim_{x \to a} x = a \quad (2.14)$$

ii. $$\lim_{x \to a} c = c \quad (2.15)$$

Example 2.13

Evaluating a Basic Limit

Evaluate each of the following limits using **Basic Limit Results**.

a. $\lim_{x \to 2} x$

b. $\lim_{x \to 2} 5$

Solution

a. The limit of x as x approaches a is a: $\lim_{x \to 2} x = 2$.

b. The limit of a constant is that constant: $\lim_{x \to 2} 5 = 5$.

We now take a look at the **limit laws**, the individual properties of limits. The proofs that these laws hold are omitted here.

Theorem 2.5: Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Let c be a constant. Then, each of the following statements holds:

Sum law for limits: $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

Difference law for limits: $\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$

Constant multiple law for limits: $\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x) = cL$

Product law for limits: $\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$

Quotient law for limits: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ for $M \neq 0$

Power law for limits: $\lim_{x \to a}(f(x))^n = \left(\lim_{x \to a} f(x)\right)^n = L^n$ for every positive integer n.

Root law for limits: $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even and $f(x) \geq 0$.

We now practice applying these limit laws to evaluate a limit.

Example 2.14

Evaluating a Limit Using Limit Laws

Use the limit laws to evaluate $\lim_{x \to -3}(4x + 2)$.

Solution

Let's apply the limit laws one step at a time to be sure we understand how they work. We need to keep in mind the requirement that, at each application of a limit law, the new limits must exist for the limit law to be applied.

$$\begin{aligned} \lim_{x \to -3}(4x + 2) &= \lim_{x \to -3} 4x + \lim_{x \to -3} 2 && \text{Apply the sum law.} \\ &= 4 \cdot \lim_{x \to -3} x + \lim_{x \to -3} 2 && \text{Apply the constant multiple law.} \\ &= 4 \cdot (-3) + 2 = -10. && \text{Apply the basic limit results and simplify.} \end{aligned}$$

Example 2.15

Using Limit Laws Repeatedly

Use the limit laws to evaluate $\lim_{x \to 2} \frac{2x^2 - 3x + 1}{x^3 + 4}$.

Solution

To find this limit, we need to apply the limit laws several times. Again, we need to keep in mind that as we rewrite the limit in terms of other limits, each new limit must exist for the limit law to be applied.

$$\begin{aligned}
\lim_{x \to 2} \frac{2x^2 - 3x + 1}{x^3 + 4} &= \frac{\lim_{x \to 2}\left(2x^2 - 3x + 1\right)}{\lim_{x \to 2}\left(x^3 + 4\right)} && \text{Apply the quotient law, making sure that. } (2)^3 + 4 \neq 0 \\
&= \frac{2 \cdot \lim_{x \to 2} x^2 - 3 \cdot \lim_{x \to 2} x + \lim_{x \to 2} 1}{\lim_{x \to 2} x^3 + \lim_{x \to 2} 4} && \text{Apply the sum law and constant multiple law.} \\
&= \frac{2 \cdot \left(\lim_{x \to 2} x\right)^2 - 3 \cdot \lim_{x \to 2} x + \lim_{x \to 2} 1}{\left(\lim_{x \to 2} x\right)^3 + \lim_{x \to 2} 4} && \text{Apply the power law.} \\
&= \frac{2(4) - 3(2) + 1}{(2)^3 + 4} = \frac{1}{4}. && \text{Apply the basic limit laws and simplify.}
\end{aligned}$$

2.11 Use the limit laws to evaluate $\lim_{x \to 6} (2x - 1)\sqrt{x + 4}$. In each step, indicate the limit law applied.

Limits of Polynomial and Rational Functions

By now you have probably noticed that, in each of the previous examples, it has been the case that $\lim_{x \to a} f(x) = f(a)$. This is not always true, but it does hold for all polynomials for any choice of *a* and for all rational functions at all values of *a* for which the rational function is defined.

Theorem 2.6: Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomial functions. Let *a* be a real number. Then,

$$\lim_{x \to a} p(x) = p(a)$$

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ when } q(a) \neq 0.$$

To see that this theorem holds, consider the polynomial $p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$. By applying the sum, constant multiple, and power laws, we end up with

$$\begin{aligned}
\lim_{x \to a} p(x) &= \lim_{x \to a}\left(c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0\right) \\
&= c_n\left(\lim_{x \to a} x\right)^n + c_{n-1}\left(\lim_{x \to a} x\right)^{n-1} + \cdots + c_1\left(\lim_{x \to a} x\right) + \lim_{x \to a} c_0 \\
&= c_n a^n + c_{n-1} a^{n-1} + \cdots + c_1 a + c_0 \\
&= p(a).
\end{aligned}$$

It now follows from the quotient law that if $p(x)$ and $q(x)$ are polynomials for which $q(a) \neq 0$, then

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}.$$

Example 2.16 applies this result.

Example 2.16

Evaluating a Limit of a Rational Function

Evaluate the $\lim\limits_{x \to 3} \frac{2x^2 - 3x + 1}{5x + 4}$.

Solution

Since 3 is in the domain of the rational function $f(x) = \frac{2x^2 - 3x + 1}{5x + 4}$, we can calculate the limit by substituting 3 for x into the function. Thus,

$$\lim_{x \to 3} \frac{2x^2 - 3x + 1}{5x + 4} = \frac{10}{19}.$$

2.12 Evaluate $\lim\limits_{x \to -2} \left(3x^3 - 2x + 7\right)$.

Additional Limit Evaluation Techniques

As we have seen, we may evaluate easily the limits of polynomials and limits of some (but not all) rational functions by direct substitution. However, as we saw in the introductory section on limits, it is certainly possible for $\lim\limits_{x \to a} f(x)$ to exist when $f(a)$ is undefined. The following observation allows us to evaluate many limits of this type:

If for all $x \neq a$, $f(x) = g(x)$ over some open interval containing a, then $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x)$.

To understand this idea better, consider the limit $\lim\limits_{x \to 1} \frac{x^2 - 1}{x - 1}$.

The function

$$\begin{aligned} f(x) &= \frac{x^2 - 1}{x - 1} \\ &= \frac{(x - 1)(x + 1)}{x - 1} \end{aligned}$$

and the function $g(x) = x + 1$ are identical for all values of $x \neq 1$. The graphs of these two functions are shown in **Figure 2.24**.

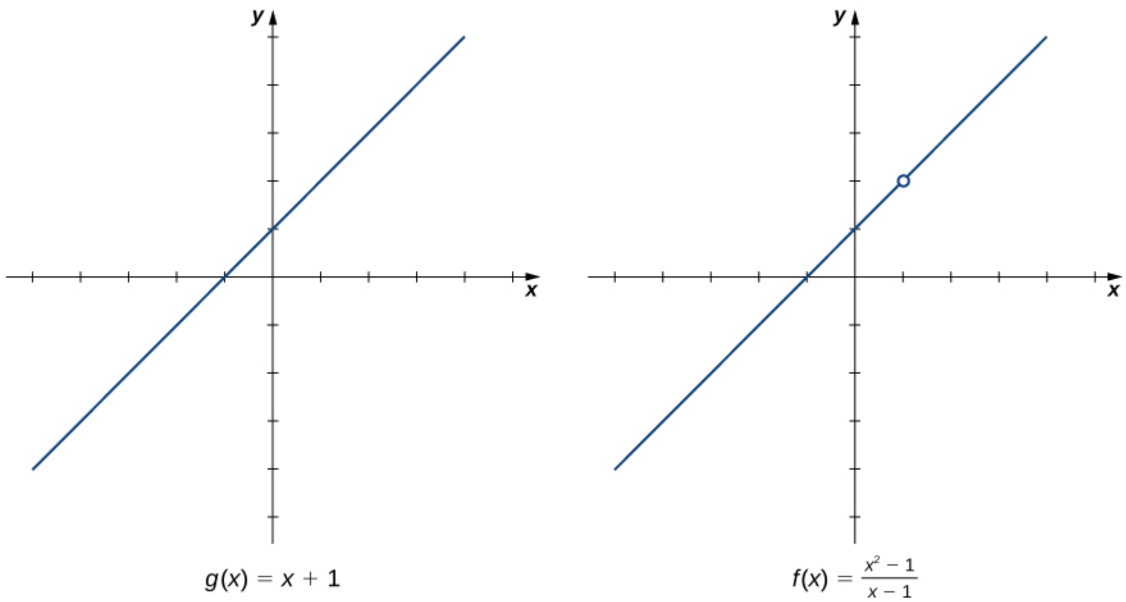


Figure 2.24 The graphs of $f(x)$ and $g(x)$ are identical for all $x \neq 1$. Their limits at 1 are equal.

We see that

$$\begin{aligned}\lim_{x \to 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1} \\ &= \lim_{x \to 1} (x + 1) \\ &= 2.\end{aligned}$$

The limit has the form $\lim_{x \to a} \frac{f(x)}{g(x)}$, where $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. (In this case, we say that $f(x)/g(x)$ has the indeterminate form $0/0$.) The following Problem-Solving Strategy provides a general outline for evaluating limits of this type.

Problem-Solving Strategy: Calculating a Limit When $f(x)/g(x)$ has the Indeterminate Form 0/0

1. First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.
2. We then need to find a function that is equal to $h(x) = f(x)/g(x)$ for all $x \neq a$ over some interval containing a. To do this, we may need to try one or more of the following steps:
   a. If $f(x)$ and $g(x)$ are polynomials, we should factor each function and cancel out any common factors.
   b. If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
   c. If $f(x)/g(x)$ is a complex fraction, we begin by simplifying it.
3. Last, we apply the limit laws.

The next examples demonstrate the use of this Problem-Solving Strategy. **Example 2.17** illustrates the factor-and-cancel technique; **Example 2.18** shows multiplying by a conjugate. In **Example 2.19**, we look at simplifying a complex fraction.

Example 2.17

Evaluating a Limit by Factoring and Canceling

Evaluate $\lim\limits_{x \to 3} \dfrac{x^2 - 3x}{2x^2 - 5x - 3}$.

Solution

Step 1. The function $f(x) = \dfrac{x^2 - 3x}{2x^2 - 5x - 3}$ is undefined for $x = 3$. In fact, if we substitute 3 into the function we get $0/0$, which is undefined. Factoring and canceling is a good strategy:

$$\lim_{x \to 3} \frac{x^2 - 3x}{2x^2 - 5x - 3} = \lim_{x \to 3} \frac{x(x-3)}{(x-3)(2x+1)}$$

Step 2. For all $x \neq 3$, $\dfrac{x^2 - 3x}{2x^2 - 5x - 3} = \dfrac{x}{2x+1}$. Therefore,

$$\lim_{x \to 3} \frac{x(x-3)}{(x-3)(2x+1)} = \lim_{x \to 3} \frac{x}{2x+1}.$$

Step 3. Evaluate using the limit laws:

$$\lim_{x \to 3} \frac{x}{2x+1} = \frac{3}{7}.$$

2.13 Evaluate $\lim\limits_{x \to -3} \dfrac{x^2 + 4x + 3}{x^2 - 9}$.

Example 2.18

Evaluating a Limit by Multiplying by a Conjugate

Evaluate $\lim\limits_{x \to -1} \dfrac{\sqrt{x+2} - 1}{x+1}$.

Solution

Step 1. $\dfrac{\sqrt{x+2} - 1}{x+1}$ has the form $0/0$ at -1. Let's begin by multiplying by $\sqrt{x+2} + 1$, the conjugate of $\sqrt{x+2} - 1$, on the numerator and denominator:

$$\lim_{x \to -1} \frac{\sqrt{x+2} - 1}{x+1} = \lim_{x \to -1} \frac{\sqrt{x+2} - 1}{x+1} \cdot \frac{\sqrt{x+2} + 1}{\sqrt{x+2} + 1}.$$

Step 2. We then multiply out the numerator. We don't multiply out the denominator because we are hoping that the $(x+1)$ in the denominator cancels out in the end:

$$= \lim_{x \to -1} \frac{x+1}{(x+1)\left(\sqrt{x+2} + 1\right)}.$$

Step 3. Then we cancel:

$$= \lim_{x \to -1} \frac{1}{\sqrt{x+2}+1}.$$

Step 4. Last, we apply the limit laws:

$$\lim_{x \to -1} \frac{1}{\sqrt{x+2}+1} = \frac{1}{2}.$$

2.14 Evaluate $\lim_{x \to 5} \frac{\sqrt{x-1}-2}{x-5}$.

Example 2.19

Evaluating a Limit by Simplifying a Complex Fraction

Evaluate $\lim_{x \to 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1}$.

Solution

Step 1. $\frac{\frac{1}{x+1} - \frac{1}{2}}{x-1}$ has the form $0/0$ at 1. We simplify the algebraic fraction by multiplying by $2(x+1)/2(x+1)$:

$$\lim_{x \to 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1} = \lim_{x \to 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1} \cdot \frac{2(x+1)}{2(x+1)}.$$

Step 2. Next, we multiply through the numerators. Do not multiply the denominators because we want to be able to cancel the factor $(x-1)$:

$$= \lim_{x \to 1} \frac{2-(x+1)}{2(x-1)(x+1)}.$$

Step 3. Then, we simplify the numerator:

$$= \lim_{x \to 1} \frac{-x+1}{2(x-1)(x+1)}.$$

Step 4. Now we factor out -1 from the numerator:

$$= \lim_{x \to 1} \frac{-(x-1)}{2(x-1)(x+1)}.$$

Step 5. Then, we cancel the common factors of $(x-1)$:

$$= \lim_{x \to 1} \frac{-1}{2(x+1)}.$$

Step 6. Last, we evaluate using the limit laws:

$$\lim_{x \to 1} \frac{-1}{2(x+1)} = -\frac{1}{4}.$$

2.15 Evaluate $\lim\limits_{x \to -3} \dfrac{\frac{1}{x+2} + 1}{x+3}$.

Example 2.20 does not fall neatly into any of the patterns established in the previous examples. However, with a little creativity, we can still use these same techniques.

Example 2.20

Evaluating a Limit When the Limit Laws Do Not Apply

Evaluate $\lim\limits_{x \to 0} \left(\dfrac{1}{x} + \dfrac{5}{x(x-5)}\right)$.

Solution

Both $1/x$ and $5/x(x-5)$ fail to have a limit at zero. Since neither of the two functions has a limit at zero, we cannot apply the sum law for limits; we must use a different strategy. In this case, we find the limit by performing addition and then applying one of our previous strategies. Observe that

$$\begin{aligned} \frac{1}{x} + \frac{5}{x(x-5)} &= \frac{x-5+5}{x(x-5)} \\ &= \frac{x}{x(x-5)}. \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{x \to 0} \left(\frac{1}{x} + \frac{5}{x(x-5)}\right) &= \lim_{x \to 0} \frac{x}{x(x-5)} \\ &= \lim_{x \to 0} \frac{1}{x-5} \\ &= -\frac{1}{5}. \end{aligned}$$

2.16 Evaluate $\lim\limits_{x \to 3} \left(\dfrac{1}{x-3} - \dfrac{4}{x^2 - 2x - 3}\right)$.

Let's now revisit one-sided limits. Simple modifications in the limit laws allow us to apply them to one-sided limits. For example, to apply the limit laws to a limit of the form $\lim\limits_{x \to a^-} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (b, a); for a limit of the form $\lim\limits_{x \to a^+} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (a, c). **Example 2.21** illustrates this point.

Example 2.21

Evaluating a One-Sided Limit Using the Limit Laws

Evaluate each of the following limits, if possible.

a. $\lim\limits_{x \to 3^-} \sqrt{x-3}$

b. $\lim_{x \to 3^+} \sqrt{x-3}$

Solution

Figure 2.25 illustrates the function $f(x) = \sqrt{x-3}$ and aids in our understanding of these limits.

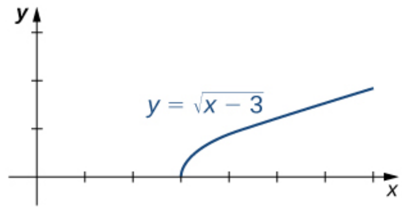


Figure 2.25 The graph shows the function $f(x) = \sqrt{x-3}$.

a. The function $f(x) = \sqrt{x-3}$ is defined over the interval $[3, +\infty)$. Since this function is not defined to the left of 3, we cannot apply the limit laws to compute $\lim_{x \to 3^-} \sqrt{x-3}$. In fact, since $f(x) = \sqrt{x-3}$ is undefined to the left of 3, $\lim_{x \to 3^-} \sqrt{x-3}$ does not exist.

b. Since $f(x) = \sqrt{x-3}$ is defined to the right of 3, the limit laws do apply to $\lim_{x \to 3^+} \sqrt{x-3}$. By applying these limit laws we obtain $\lim_{x \to 3^+} \sqrt{x-3} = 0$.

In **Example 2.22** we look at one-sided limits of a piecewise-defined function and use these limits to draw a conclusion about a two-sided limit of the same function.

Example 2.22

Evaluating a Two-Sided Limit Using the Limit Laws

For $f(x) = \begin{cases} 4x-3 & \text{if } x < 2 \\ (x-3)^2 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits:

a. $\lim_{x \to 2^-} f(x)$

b. $\lim_{x \to 2^+} f(x)$

c. $\lim_{x \to 2} f(x)$

Solution

Figure 2.26 illustrates the function $f(x)$ and aids in our understanding of these limits.

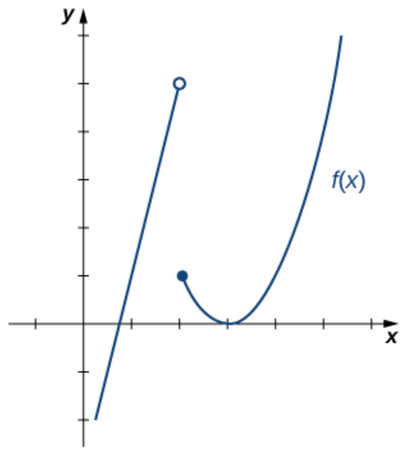


Figure 2.26 This graph shows a function $f(x)$.

a. Since $f(x) = 4x - 3$ for all x in $(-\infty, 2)$, replace $f(x)$ in the limit with $4x - 3$ and apply the limit laws:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4x - 3) = 5.$$

b. Since $f(x) = (x - 3)^2$ for all x in $(2, +\infty)$, replace $f(x)$ in the limit with $(x - 3)^2$ and apply the limit laws:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x - 3)^2 = 1.$$

c. Since $\lim_{x \to 2^-} f(x) = 5$ and $\lim_{x \to 2^+} f(x) = 1$, we conclude that $\lim_{x \to 2} f(x)$ does not exist.

2.17 Graph $f(x) = \begin{cases} -x - 2 \text{ if } x < -1 \\ 2 \text{ if } x = -1 \\ x^3 \text{ if } x > -1 \end{cases}$ and evaluate $\lim_{x \to -1^-} f(x)$.

We now turn our attention to evaluating a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$, where $\lim_{x \to a} f(x) = K$, where $K \neq 0$ and $\lim_{x \to a} g(x) = 0$. That is, $f(x)/g(x)$ has the form $K/0, K \neq 0$ at a.

Example 2.23

Evaluating a Limit of the Form $K/0, K \neq 0$ Using the Limit Laws

Evaluate $\lim_{x \to 2^-} \frac{x - 3}{x^2 - 2x}$.

Solution

Step 1. After substituting in $x = 2$, we see that this limit has the form $-1/0$. That is, as x approaches 2 from the left, the numerator approaches −1; and the denominator approaches 0. Consequently, the magnitude of $\frac{x-3}{x(x-2)}$ becomes infinite. To get a better idea of what the limit is, we need to factor the denominator:

$$\lim_{x \to 2^-} \frac{x-3}{x^2-2x} = \lim_{x \to 2^-} \frac{x-3}{x(x-2)}.$$

Step 2. Since $x-2$ is the only part of the denominator that is zero when 2 is substituted, we then separate $1/(x-2)$ from the rest of the function:

$$= \lim_{x \to 2^-} \frac{x-3}{x} \cdot \frac{1}{x-2}.$$

Step 3. $\lim_{x \to 2^-} \frac{x-3}{x} = -\frac{1}{2}$ and $\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$. Therefore, the product of $(x-3)/x$ and $1/(x-2)$ has a limit of $+\infty$:

$$\lim_{x \to 2^-} \frac{x-3}{x^2-2x} = +\infty.$$

2.18 Evaluate $\lim_{x \to 1} \frac{x+2}{(x-1)^2}$.

The Squeeze Theorem

The techniques we have developed thus far work very well for algebraic functions, but we are still unable to evaluate limits of very basic trigonometric functions. The next theorem, called the **squeeze theorem**, proves very useful for establishing basic trigonometric limits. This theorem allows us to calculate limits by “squeezing” a function, with a limit at a point *a* that is unknown, between two functions having a common known limit at *a*. Figure 2.27 illustrates this idea.

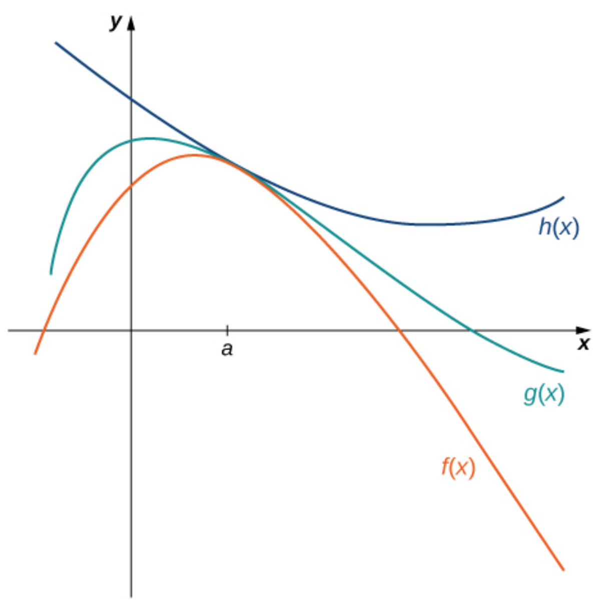


Figure 2.27 The Squeeze Theorem applies when $f(x) \le g(x) \le h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$.

Theorem 2.7: The Squeeze Theorem

Let $f(x)$, $g(x)$, and $h(x)$ be defined for all $x \neq a$ over an open interval containing a. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where L is a real number, then $\lim_{x \to a} g(x) = L$.

Example 2.24

Applying the Squeeze Theorem

Apply the squeeze theorem to evaluate $\lim_{x \to 0} x\cos x$.

Solution

Because $-1 \leq \cos x \leq 1$ for all x, we have $-|x| \leq x\cos x \leq |x|$. Since $\lim_{x \to 0}(-|x|) = 0 = \lim_{x \to 0}|x|$, from the squeeze theorem, we obtain $\lim_{x \to 0} x\cos x = 0$. The graphs of $f(x) = -|x|$, $g(x) = x\cos x$, and $h(x) = |x|$ are shown in Figure 2.28.

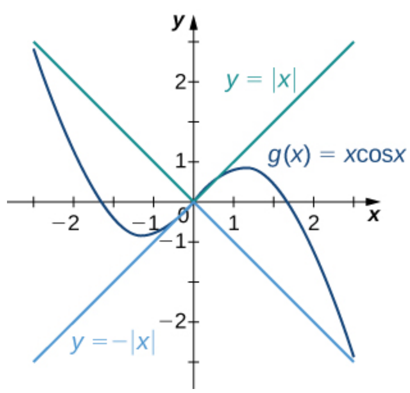


Figure 2.28 The graphs of $f(x)$, $g(x)$, and $h(x)$ are shown around the point $x = 0$.

2.19 Use the squeeze theorem to evaluate $\lim_{x \to 0} x^2 \sin\frac{1}{x}$.

We now use the squeeze theorem to tackle several very important limits. Although this discussion is somewhat lengthy, these limits prove invaluable for the development of the material in both the next section and the next chapter. The first of these limits is $\lim_{\theta \to 0} \sin\theta$. Consider the unit circle shown in Figure 2.29. In the figure, we see that $\sin\theta$ is the y-coordinate on the unit circle and it corresponds to the line segment shown in blue. The radian measure of angle θ is the length of the arc it subtends on the unit circle. Therefore, we see that for $0 < \theta < \frac{\pi}{2}$, $0 < \sin\theta < \theta$.

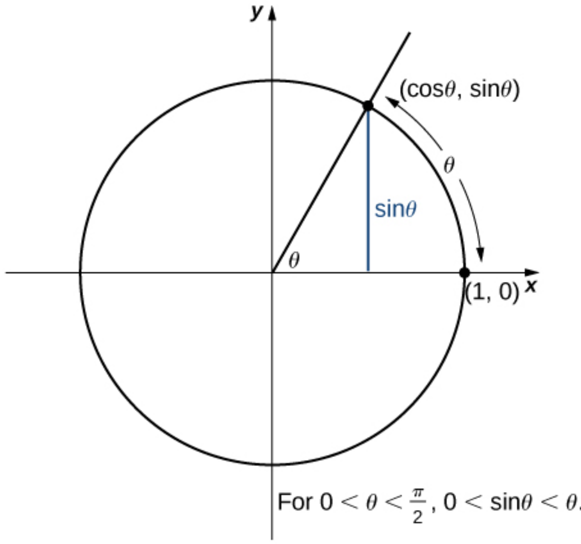


Figure 2.29 The sine function is shown as a line on the unit circle.

Because $\lim\limits_{\theta \to 0^+} 0 = 0$ and $\lim\limits_{\theta \to 0^+} \theta = 0,$ by using the squeeze theorem we conclude that

$$\lim_{\theta \to 0^+} \sin\theta = 0.$$

To see that $\lim\limits_{\theta \to 0^-} \sin\theta = 0$ as well, observe that for $-\frac{\pi}{2} < \theta < 0,\ 0 < -\theta < \frac{\pi}{2}$ and hence, $0 < \sin(-\theta) < -\theta.$ Consequently, $0 < -\sin\theta < -\theta.$ It follows that $0 > \sin\theta > \theta.$ An application of the squeeze theorem produces the desired limit. Thus, since $\lim\limits_{\theta \to 0^+} \sin\theta = 0$ and $\lim\limits_{\theta \to 0^-} \sin\theta = 0,$

$$\lim_{\theta \to 0} \sin\theta = 0. \tag{2.16}$$

Next, using the identity $\cos\theta = \sqrt{1 - \sin^2\theta}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2},$ we see that

$$\lim_{\theta \to 0} \cos\theta = \lim_{\theta \to 0} \sqrt{1 - \sin^2\theta} = 1. \tag{2.17}$$

We now take a look at a limit that plays an important role in later chapters—namely, $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta}.$ To evaluate this limit, we use the unit circle in **Figure 2.30**. Notice that this figure adds one additional triangle to **Figure 2.30**. We see that the length of the side opposite angle θ in this new triangle is $\tan\theta.$ Thus, we see that for $0 < \theta < \frac{\pi}{2},\ \sin\theta < \theta < \tan\theta.$

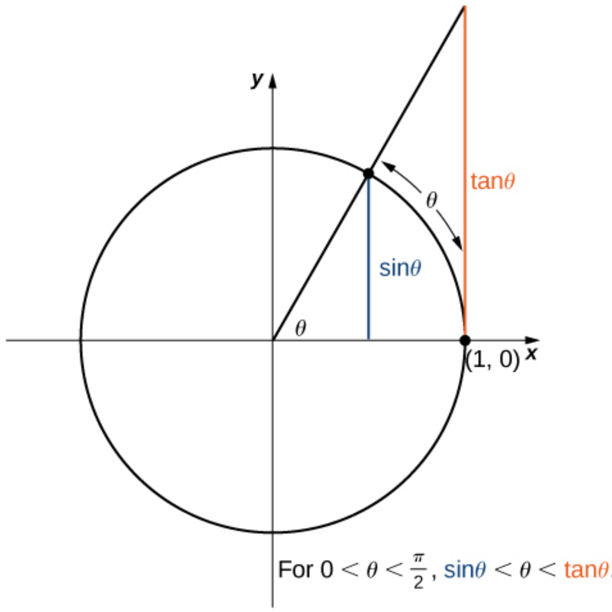


Figure 2.30 The sine and tangent functions are shown as lines on the unit circle.

By dividing by $\sin\theta$ in all parts of the inequality, we obtain

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Equivalently, we have

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim\limits_{\theta \to 0^+} 1 = 1 = \lim\limits_{\theta \to 0^+} \cos\theta$, we conclude that $\lim\limits_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$. By applying a manipulation similar to that used in demonstrating that $\lim\limits_{\theta \to 0^-} \sin\theta = 0$, we can show that $\lim\limits_{\theta \to 0^-} \frac{\sin\theta}{\theta} = 1$. Thus,

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1. \tag{2.18}$$

In **Example 2.25** we use this limit to establish $\lim\limits_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$. This limit also proves useful in later chapters.

Example 2.25

Evaluating an Important Trigonometric Limit

Evaluate $\lim\limits_{\theta \to 0} \frac{1 - \cos\theta}{\theta}$.

Solution

In the first step, we multiply by the conjugate so that we can use a trigonometric identity to convert the cosine in the numerator to a sine:

$$\begin{aligned}\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} &= \lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} \cdot \frac{1 + \cos\theta}{1 + \cos\theta} \\ &= \lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\theta(1 + \cos\theta)} \\ &= \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(1 + \cos\theta)} \\ &= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1 + \cos\theta} \\ &= 1 \cdot \frac{0}{2} = 0.\end{aligned}$$

Therefore,

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0. \tag{2.19}$$

2.20 Evaluate $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\sin\theta}$.

Student PROJECT

Deriving the Formula for the Area of a Circle

Some of the geometric formulas we take for granted today were first derived by methods that anticipate some of the methods of calculus. The Greek mathematician Archimedes (ca. 287−212; BCE) was particularly inventive, using polygons inscribed within circles to approximate the area of the circle as the number of sides of the polygon increased. He never came up with the idea of a limit, but we can use this idea to see what his geometric constructions could have predicted about the limit.

We can estimate the area of a circle by computing the area of an inscribed regular polygon. Think of the regular polygon as being made up of n triangles. By taking the limit as the vertex angle of these triangles goes to zero, you can obtain the area of the circle. To see this, carry out the following steps:

1. Express the height h and the base b of the isosceles triangle in **Figure 2.31** in terms of θ and r.

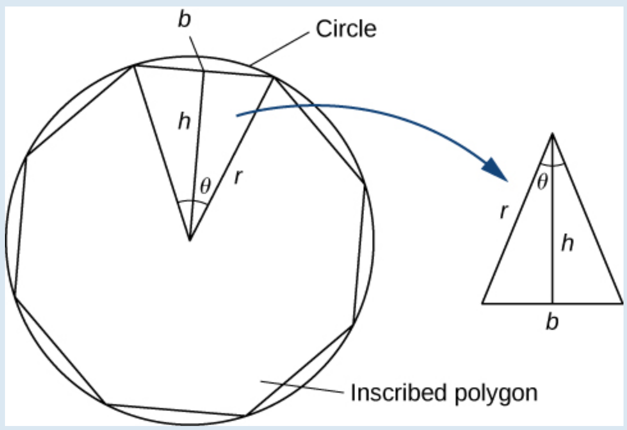


Figure 2.31

2. Using the expressions that you obtained in step 1, express the area of the isosceles triangle in terms of θ and r. (Substitute $(1/2)\sin\theta$ for $\sin(\theta/2)\cos(\theta/2)$ in your expression.)
3. If an n-sided regular polygon is inscribed in a circle of radius r, find a relationship between θ and n. Solve this for n. Keep in mind there are 2π radians in a circle. (Use radians, not degrees.)
4. Find an expression for the area of the n-sided polygon in terms of r and θ.
5. To find a formula for the area of the circle, find the limit of the expression in step 4 as θ goes to zero. (*Hint:* $\lim\limits_{\theta \to 0} \frac{(\sin\theta)}{\theta} = 1$).

The technique of estimating areas of regions by using polygons is revisited in **Introduction to Integration**.

2.3 EXERCISES

In the following exercises, use the limit laws to evaluate each limit. Justify each step by indicating the appropriate limit law(s).

83. $\lim_{x \to 0} \left(4x^2 - 2x + 3\right)$

84. $\lim_{x \to 1} \frac{x^3 + 3x^2 + 5}{4 - 7x}$

85. $\lim_{x \to -2} \sqrt{x^2 - 6x + 3}$

86. $\lim_{x \to -1} (9x + 1)^2$

In the following exercises, use direct substitution to evaluate each limit.

87. $\lim_{x \to 7} x^2$

88. $\lim_{x \to -2} \left(4x^2 - 1\right)$

89. $\lim_{x \to 0} \frac{1}{1 + \sin x}$

90. $\lim_{x \to 2} e^{2x - x^2}$

91. $\lim_{x \to 1} \frac{2 - 7x}{x + 6}$

92. $\lim_{x \to 3} \ln e^{3x}$

In the following exercises, use direct substitution to show that each limit leads to the indeterminate form $0/0$. Then, evaluate the limit.

93. $\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$

94. $\lim_{x \to 2} \frac{x - 2}{x^2 - 2x}$

95. $\lim_{x \to 6} \frac{3x - 18}{2x - 12}$

96. $\lim_{h \to 0} \frac{(1 + h)^2 - 1}{h}$

97. $\lim_{t \to 9} \frac{t - 9}{\sqrt{t} - 3}$

98. $\lim_{h \to 0} \frac{\frac{1}{a + h} - \frac{1}{a}}{h}$, where a is a non-zero real-valued constant

99. $\lim_{\theta \to \pi} \frac{\sin \theta}{\tan \theta}$

100. $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1}$

101. $\lim_{x \to 1/2} \frac{2x^2 + 3x - 2}{2x - 1}$

102. $\lim_{x \to -3} \frac{\sqrt{x + 4} - 1}{x + 3}$

In the following exercises, use direct substitution to obtain an undefined expression. Then, use the method of **Example 2.23** to simplify the function to help determine the limit.

103. $\lim_{x \to -2^-} \frac{2x^2 + 7x - 4}{x^2 + x - 2}$

104. $\lim_{x \to -2^+} \frac{2x^2 + 7x - 4}{x^2 + x - 2}$

105. $\lim_{x \to 1^-} \frac{2x^2 + 7x - 4}{x^2 + x - 2}$

106. $\lim_{x \to 1^+} \frac{2x^2 + 7x - 4}{x^2 + x - 2}$

In the following exercises, assume that $\lim_{x \to 6} f(x) = 4$, $\lim_{x \to 6} g(x) = 9$, and $\lim_{x \to 6} h(x) = 6$. Use these three facts and the limit laws to evaluate each limit.

107. $\lim_{x \to 6} 2f(x)g(x)$

108. $\lim_{x \to 6} \frac{g(x) - 1}{f(x)}$

109. $\lim_{x \to 6} \left(f(x) + \frac{1}{3}g(x)\right)$

110. $\lim_{x \to 6} \frac{(h(x))^3}{2}$

111. $\lim_{x \to 6} \sqrt{g(x) - f(x)}$

112. $\lim_{x \to 6} x \cdot h(x)$

113. $\lim\limits_{x \to 6} [(x+1) \cdot f(x)]$

114. $\lim\limits_{x \to 6} (f(x) \cdot g(x) - h(x))$

[T] In the following exercises, use a calculator to draw the graph of each piecewise-defined function and study the graph to evaluate the given limits.

115. $f(x) = \begin{cases} x^2, & x \le 3 \\ x+4, & x > 3 \end{cases}$

a. $\lim\limits_{x \to 3^-} f(x)$

b. $\lim\limits_{x \to 3^+} f(x)$

116. $g(x) = \begin{cases} x^3 - 1, & x \le 0 \\ 1, & x > 0 \end{cases}$

a. $\lim\limits_{x \to 0^-} g(x)$

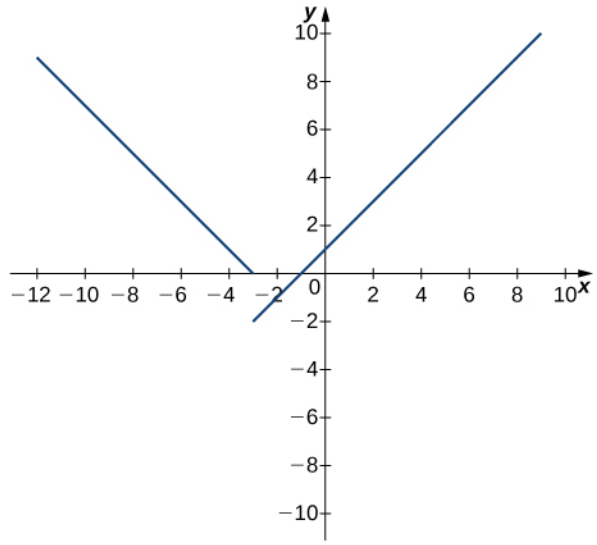
b. $\lim\limits_{x \to 0^+} g(x)$

117. $h(x) = \begin{cases} x^2 - 2x + 1, & x < 2 \\ 3 - x, & x \ge 2 \end{cases}$

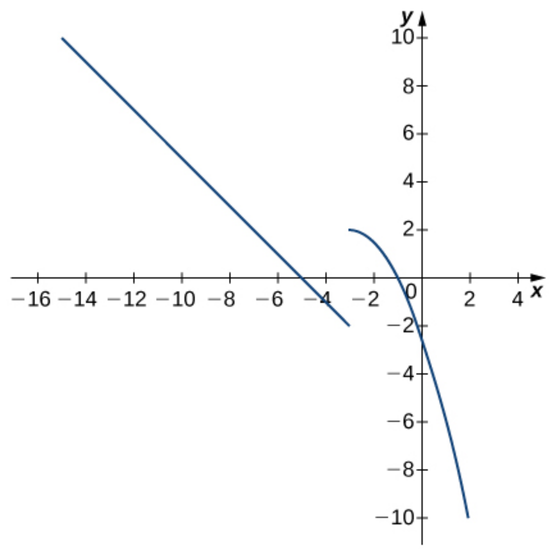
a. $\lim\limits_{x \to 2^-} h(x)$

b. $\lim\limits_{x \to 2^+} h(x)$

In the following exercises, use the following graphs and the limit laws to evaluate each limit.



$y = f(x)$



$y = g(x)$

118. $\lim\limits_{x \to -3^+} (f(x) + g(x))$

119. $\lim\limits_{x \to -3^-} (f(x) - 3g(x))$

120. $\lim\limits_{x \to 0} \dfrac{f(x)g(x)}{3}$

121. $\lim\limits_{x \to -5} \dfrac{2 + g(x)}{f(x)}$

122. $\lim\limits_{x \to 1} (f(x))^2$

123. $\lim\limits_{x \to 1} \sqrt[3]{f(x) - g(x)}$

124. $\lim_{x \to -7}(x \cdot g(x))$

125. $\lim_{x \to -9}[x \cdot f(x) + 2 \cdot g(x)]$

126. **[T]** True or False? If $2x - 1 \le g(x) \le x^2 - 2x + 3,$ then $\lim_{x \to 2} g(x) = 0.$

For the following problems, evaluate the limit using the squeeze theorem. Use a calculator to graph the functions $f(x),\ g(x),$ and $h(x)$ when possible.

127. **[T]** $\lim_{\theta \to 0} \theta^2 \cos\left(\frac{1}{\theta}\right)$

128. $\lim_{x \to 0} f(x),$ where $f(x) = \begin{cases} 0, & x \text{ rational} \\ x^2, & x \text{ irrrational} \end{cases}$

129. **[T]** In physics, the magnitude of an electric field generated by a point charge at a distance r in vacuum is governed by Coulomb's law: $E(r) = \frac{q}{4\pi\varepsilon_0 r^2},$ where E represents the magnitude of the electric field, q is the charge of the particle, r is the distance between the particle and where the strength of the field is measured, and $\frac{1}{4\pi\varepsilon_0}$ is Coulomb's constant: $8.988 \times 10^9\ \text{N} \cdot \text{m}^2/\text{C}^2.$

a. Use a graphing calculator to graph $E(r)$ given that the charge of the particle is $q = 10^{-10}.$
b. Evaluate $\lim_{r \to 0^+} E(r).$ What is the physical meaning of this quantity? Is it physically relevant? Why are you evaluating from the right?

130. **[T]** The density of an object is given by its mass divided by its volume: $\rho = m/V.$

a. Use a calculator to plot the volume as a function of density $(V = m/\rho),$ assuming you are examining something of mass 8 kg ($m = 8$).
b. Evaluate $\lim_{\rho \to 0^+} V(\rho)$ and explain the physical meaning.

2.4 | Continuity

Learning Objectives

2.4.1 Explain the three conditions for continuity at a point.
2.4.2 Describe three kinds of discontinuities.
2.4.3 Define continuity on an interval.
2.4.4 State the theorem for limits of composite functions.
2.4.5 Provide an example of the intermediate value theorem.

Many functions have the property that their graphs can be traced with a pencil without lifting the pencil from the page. Such functions are called *continuous*. Other functions have points at which a break in the graph occurs, but satisfy this property over intervals contained in their domains. They are continuous on these intervals and are said to have a *discontinuity at a point* where a break occurs.

We begin our investigation of continuity by exploring what it means for a function to have *continuity at a point*. Intuitively, a function is continuous at a particular point if there is no break in its graph at that point.

Continuity at a Point

Before we look at a formal definition of what it means for a function to be continuous at a point, let's consider various functions that fail to meet our intuitive notion of what it means to be continuous at a point. We then create a list of conditions that prevent such failures.

Our first function of interest is shown in **Figure 2.32**. We see that the graph of $f(x)$ has a hole at a. In fact, $f(a)$ is undefined. At the very least, for $f(x)$ to be continuous at a, we need the following condition:

i. $f(a)$ is defined.

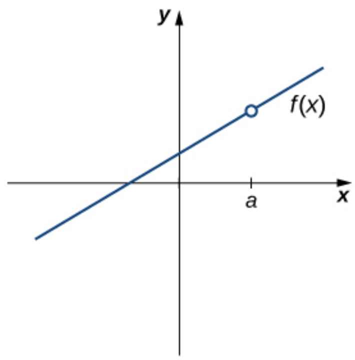


Figure 2.32 The function $f(x)$ is not continuous at a because $f(a)$ is undefined.

However, as we see in **Figure 2.33**, this condition alone is insufficient to guarantee continuity at the point a. Although $f(a)$ is defined, the function has a gap at a. In this example, the gap exists because $\lim_{x \to a} f(x)$ does not exist. We must add another condition for continuity at a—namely,

ii. $\lim_{x \to a} f(x)$ exists.

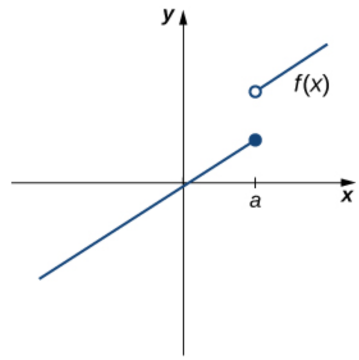


Figure 2.33 The function $f(x)$ is not continuous at a because $\lim\limits_{x \to a} f(x)$ does not exist.

However, as we see in **Figure 2.34**, these two conditions by themselves do not guarantee continuity at a point. The function in this figure satisfies both of our first two conditions, but is still not continuous at a. We must add a third condition to our list:

iii. $\lim\limits_{x \to a} f(x) = f(a)$.

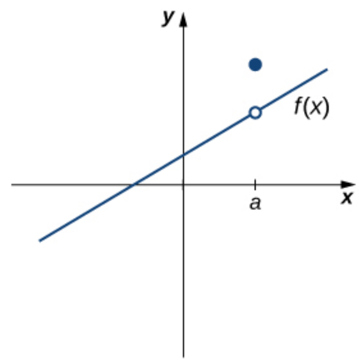


Figure 2.34 The function $f(x)$ is not continuous at a because $\lim\limits_{x \to a} f(x) \neq f(a)$.

Now we put our list of conditions together and form a definition of continuity at a point.

Definition

A function $f(x)$ is **continuous at a point** a if and only if the following three conditions are satisfied:

i. $f(a)$ is defined

ii. $\lim\limits_{x \to a} f(x)$ exists

iii. $\lim\limits_{x \to a} f(x) = f(a)$

A function is **discontinuous at a point** a if it fails to be continuous at a.

The following procedure can be used to analyze the continuity of a function at a point using this definition.

Problem-Solving Strategy: Determining Continuity at a Point

1. Check to see if $f(a)$ is defined. If $f(a)$ is undefined, we need go no further. The function is not continuous at a. If $f(a)$ is defined, continue to step 2.
2. Compute $\lim\limits_{x \to a} f(x)$. In some cases, we may need to do this by first computing $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$. If $\lim\limits_{x \to a} f(x)$ does not exist (that is, it is not a real number), then the function is not continuous at a and the problem is solved. If $\lim\limits_{x \to a} f(x)$ exists, then continue to step 3.
3. Compare $f(a)$ and $\lim\limits_{x \to a} f(x)$. If $\lim\limits_{x \to a} f(x) \neq f(a)$, then the function is not continuous at a. If $\lim\limits_{x \to a} f(x) = f(a)$, then the function is continuous at a.

The next three examples demonstrate how to apply this definition to determine whether a function is continuous at a given point. These examples illustrate situations in which each of the conditions for continuity in the definition succeed or fail.

Example 2.26

Determining Continuity at a Point, Condition 1

Using the definition, determine whether the function $f(x) = (x^2 - 4)/(x - 2)$ is continuous at $x = 2$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(2)$. We can see that $f(2) = 0/0$, which is undefined. Therefore, $f(x) = \frac{x^2 - 4}{x - 2}$ is discontinuous at 2 because $f(2)$ is undefined. The graph of $f(x)$ is shown in Figure 2.35.

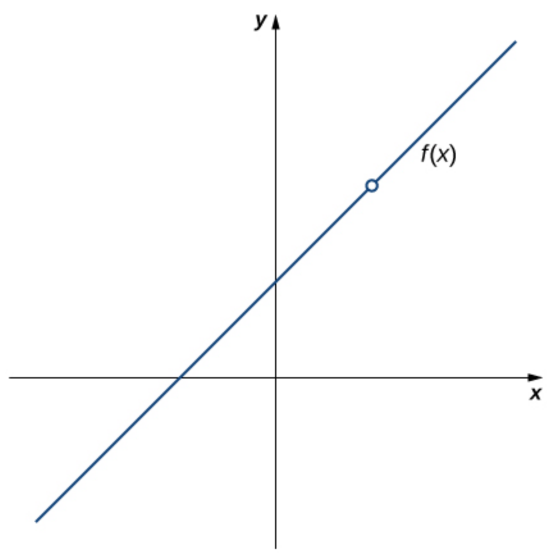


Figure 2.35 The function $f(x)$ is discontinuous at 2 because $f(2)$ is undefined.

Example 2.27

Determining Continuity at a Point, Condition 2

Using the definition, determine whether the function $f(x) = \begin{cases} -x^2 + 4 & \text{if } x \le 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(3)$.

$$f(3) = -(3^2) + 4 = -5.$$

Thus, $f(3)$ is defined. Next, we calculate $\lim_{x \to 3} f(x)$. To do this, we must compute $\lim_{x \to 3^-} f(x)$ and $\lim_{x \to 3^+} f(x)$:

$$\lim_{x \to 3^-} f(x) = -(3^2) + 4 = -5$$

and

$$\lim_{x \to 3^+} f(x) = 4(3) - 8 = 4.$$

Therefore, $\lim_{x \to 3} f(x)$ does not exist. Thus, $f(x)$ is not continuous at 3. The graph of $f(x)$ is shown in **Figure 2.36**.

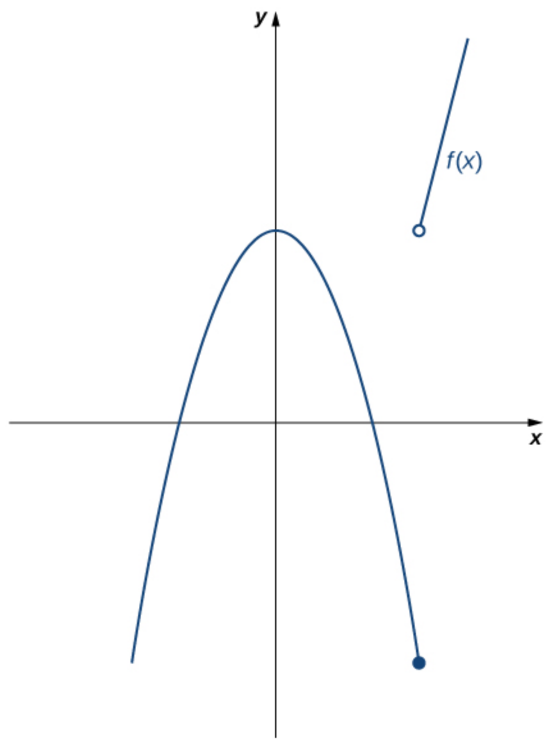


Figure 2.36 The function $f(x)$ is not continuous at 3 because $\lim_{x \to 3} f(x)$ does not exist.

Example 2.28

Determining Continuity at a Point, Condition 3

Using the definition, determine whether the function $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.

Solution

First, observe that

$$f(0) = 1.$$

Next,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Last, compare $f(0)$ and $\lim_{x \to 1} f(x)$. We see that

$$f(0) = 1 = \lim_{x \to 0} f(x).$$

Since all three of the conditions in the definition of continuity are satisfied, $f(x)$ is continuous at $x = 0$.

2.21 Using the definition, determine whether the function $f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ -x + 4 & \text{if } x > 1 \end{cases}$ is continuous at $x = 1$.

If the function is not continuous at 1, indicate the condition for continuity at a point that fails to hold.

By applying the definition of continuity and previously established theorems concerning the evaluation of limits, we can state the following theorem.

Theorem 2.8: Continuity of Polynomials and Rational Functions

Polynomials and rational functions are continuous at every point in their domains.

Proof

Previously, we showed that if $p(x)$ and $q(x)$ are polynomials, $\lim_{x \to a} p(x) = p(a)$ for every polynomial $p(x)$ and $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ as long as $q(a) \neq 0$. Therefore, polynomials and rational functions are continuous on their domains.

□

We now apply **Continuity of Polynomials and Rational Functions** to determine the points at which a given rational function is continuous.

Example 2.29

Continuity of a Rational Function

For what values of x is $f(x) = \frac{x+1}{x-5}$ continuous?

Solution

The rational function $f(x) = \frac{x+1}{x-5}$ is continuous for every value of x except $x = 5$.

2.22 For what values of x is $f(x) = 3x^4 - 4x^2$ continuous?

Types of Discontinuities

As we have seen in **Example 2.26** and **Example 2.27**, discontinuities take on several different appearances. We classify the types of discontinuities we have seen thus far as removable discontinuities, infinite discontinuities, or jump discontinuities. Intuitively, a **removable discontinuity** is a discontinuity for which there is a hole in the graph, a **jump discontinuity** is a noninfinite discontinuity for which the sections of the function do not meet up, and an **infinite discontinuity** is a discontinuity located at a vertical asymptote. **Figure 2.37** illustrates the differences in these types of discontinuities. Although these terms provide a handy way of describing three common types of discontinuities, keep in mind that not all discontinuities fit neatly into these categories.

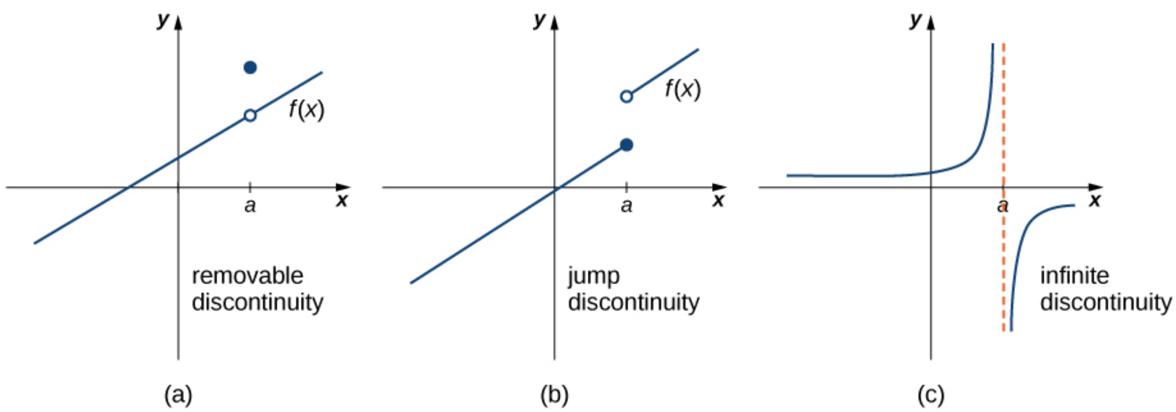


Figure 2.37 Discontinuities are classified as (a) removable, (b) jump, or (c) infinite.

These three discontinuities are formally defined as follows:

Definition

If $f(x)$ is discontinuous at a, then

1. f has a **removable discontinuity** at a if $\lim_{x \to a} f(x)$ exists. (Note: When we state that $\lim_{x \to a} f(x)$ exists, we mean that $\lim_{x \to a} f(x) = L$, where L is a real number.)
2. f has a **jump discontinuity** at a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$. (Note: When we state that $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)
3. f has an **infinite discontinuity** at a if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

Example 2.30

Classifying a Discontinuity

In **Example 2.26**, we showed that $f(x) = \frac{x^2 - 4}{x - 2}$ is discontinuous at $x = 2.$ Classify this discontinuity as removable, jump, or infinite.

Solution

To classify the discontinuity at 2 we must evaluate $\lim_{x \to 2} f(x)$:

$$\begin{aligned}\lim_{x \to 2} f(x) &= \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} \\ &= \lim_{x \to 2} (x+2) \\ &= 4.\end{aligned}$$

Since f is discontinuous at 2 and $\lim_{x \to 2} f(x)$ exists, f has a removable discontinuity at $x = 2.$

Example 2.31

Classifying a Discontinuity

In **Example 2.27**, we showed that $f(x) = \begin{cases} -x^2 + 4 & \text{if } x \le 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$ is discontinuous at $x = 3.$ Classify this discontinuity as removable, jump, or infinite.

Solution

Earlier, we showed that f is discontinuous at 3 because $\lim_{x \to 3} f(x)$ does not exist. However, since $\lim_{x \to 3^-} f(x) = -5$ and $\lim_{x \to 3^+} f(x) = 4$ both exist, we conclude that the function has a jump discontinuity at 3.

Example 2.32

Classifying a Discontinuity

Determine whether $f(x) = \frac{x + 2}{x + 1}$ is continuous at $-1.$ If the function is discontinuous at $-1,$ classify the discontinuity as removable, jump, or infinite.

Solution

The function value $f(-1)$ is undefined. Therefore, the function is not continuous at $-1.$ To determine the type of discontinuity, we must determine the limit at -1. We see that $\lim_{x \to -1^-} \frac{x+2}{x+1} = -\infty$ and $\lim_{x \to -1^+} \frac{x+2}{x+1} = +\infty$. Therefore, the function has an infinite discontinuity at -1.

2.23 For $f(x) = \begin{cases} x^2 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$, decide whether f is continuous at 1. If f is not continuous at 1, classify the discontinuity as removable, jump, or infinite.

Continuity over an Interval

Now that we have explored the concept of continuity at a point, we extend that idea to **continuity over an interval**. As we develop this idea for different types of intervals, it may be useful to keep in mind the intuitive idea that a function is continuous over an interval if we can use a pencil to trace the function between any two points in the interval without lifting the pencil from the paper. In preparation for defining continuity on an interval, we begin by looking at the definition of what it means for a function to be continuous from the right at a point and continuous from the left at a point.

Continuity from the Right and from the Left

A function $f(x)$ is said to be **continuous from the right** at a if $\lim_{x \to a^+} f(x) = f(a)$.

A function $f(x)$ is said to be **continuous from the left** at a if $\lim_{x \to a^-} f(x) = f(a)$.

A function is continuous over an open interval if it is continuous at every point in the interval. A function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b) and is continuous from the right at a and is continuous from the left at b. Analogously, a function $f(x)$ is continuous over an interval of the form $(a, b]$ if it is continuous over (a, b) and is continuous from the left at b. Continuity over other types of intervals are defined in a similar fashion.

Requiring that $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$ ensures that we can trace the graph of the function from the point $(a, f(a))$ to the point $(b, f(b))$ without lifting the pencil. If, for example, $\lim_{x \to a^+} f(x) \neq f(a)$, we would need to lift our pencil to jump from $f(a)$ to the graph of the rest of the function over $(a, b]$.

Example 2.33

Continuity on an Interval

State the interval(s) over which the function $f(x) = \frac{x-1}{x^2+2x}$ is continuous.

Solution

Since $f(x) = \frac{x-1}{x^2+2x}$ is a rational function, it is continuous at every point in its domain. The domain of $f(x)$ is the set $(-\infty, -2) \cup (-2, 0) \cup (0, +\infty)$. Thus, $f(x)$ is continuous over each of the intervals

$$(-\infty, -2), (-2, 0), \text{ and } (0, +\infty).$$

Example 2.34

Continuity over an Interval

State the interval(s) over which the function $f(x) = \sqrt{4 - x^2}$ is continuous.

Solution

From the limit laws, we know that $\lim\limits_{x \to a} \sqrt{4 - x^2} = \sqrt{4 - a^2}$ for all values of a in $(-2, 2)$. We also know that $\lim\limits_{x \to -2^+} \sqrt{4 - x^2} = 0$ exists and $\lim\limits_{x \to 2^-} \sqrt{4 - x^2} = 0$ exists. Therefore, $f(x)$ is continuous over the interval $[-2, 2]$.

2.24 State the interval(s) over which the function $f(x) = \sqrt{x + 3}$ is continuous.

The **Composite Function Theorem** allows us to expand our ability to compute limits. In particular, this theorem ultimately allows us to demonstrate that trigonometric functions are continuous over their domains.

Theorem 2.9: Composite Function Theorem

If $f(x)$ is continuous at L and $\lim\limits_{x \to a} g(x) = L$, then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L).$$

Before we move on to **Example 2.35**, recall that earlier, in the section on limit laws, we showed $\lim\limits_{x \to 0} \cos x = 1 = \cos(0)$. Consequently, we know that $f(x) = \cos x$ is continuous at 0. In **Example 2.35** we see how to combine this result with the composite function theorem.

Example 2.35

Limit of a Composite Cosine Function

Evaluate $\lim\limits_{x \to \pi/2} \cos\left(x - \frac{\pi}{2}\right)$.

Solution

The given function is a composite of $\cos x$ and $x - \frac{\pi}{2}$. Since $\lim\limits_{x \to \pi/2}\left(x - \frac{\pi}{2}\right) = 0$ and $\cos x$ is continuous at 0, we may apply the composite function theorem. Thus,

$$\lim_{x \to \pi/2} \cos\left(x - \frac{\pi}{2}\right) = \cos\left(\lim_{x \to \pi/2}\left(x - \frac{\pi}{2}\right)\right) = \cos(0) = 1.$$

2.25 Evaluate $\lim\limits_{x \to \pi} \sin(x - \pi)$.

The proof of the next theorem uses the composite function theorem as well as the continuity of $f(x) = \sin x$ and $g(x) = \cos x$ at the point 0 to show that trigonometric functions are continuous over their entire domains.

Theorem 2.10: Continuity of Trigonometric Functions

Trigonometric functions are continuous over their entire domains.

Proof

We begin by demonstrating that $\cos x$ is continuous at every real number. To do this, we must show that $\lim\limits_{x \to a} \cos x = \cos a$ for all values of a.

$$\begin{aligned}
\lim_{x \to a} \cos x &= \lim_{x \to a} \cos((x - a) + a) && \text{rewrite } x = x - a + a \\
&= \lim_{x \to a} (\cos(x - a)\cos a - \sin(x - a)\sin a) && \text{apply the identity for the cosine of the sum of two angles} \\
&= \cos\left(\lim_{x \to a}(x - a)\right)\cos a - \sin\left(\lim_{x \to a}(x - a)\right)\sin a && \lim_{x \to a}(x - a) = 0, \text{ and } \sin x \text{ and } \cos x \text{ are continuous at } 0 \\
&= \cos(0)\cos a - \sin(0)\sin a && \text{evaluate } \cos(0) \text{ and } \sin(0) \text{ and simplify} \\
&= 1 \cdot \cos a - 0 \cdot \sin a = \cos a.
\end{aligned}$$

The proof that $\sin x$ is continuous at every real number is analogous. Because the remaining trigonometric functions may be expressed in terms of $\sin x$ and $\cos x$, their continuity follows from the quotient limit law.

□

As you can see, the composite function theorem is invaluable in demonstrating the continuity of trigonometric functions. As we continue our study of calculus, we revisit this theorem many times.

The Intermediate Value Theorem

Functions that are continuous over intervals of the form $[a, b]$, where a and b are real numbers, exhibit many useful properties. Throughout our study of calculus, we will encounter many powerful theorems concerning such functions. The first of these theorems is the **Intermediate Value Theorem**.

Theorem 2.11: The Intermediate Value Theorem

Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$ in **Figure 2.38**.

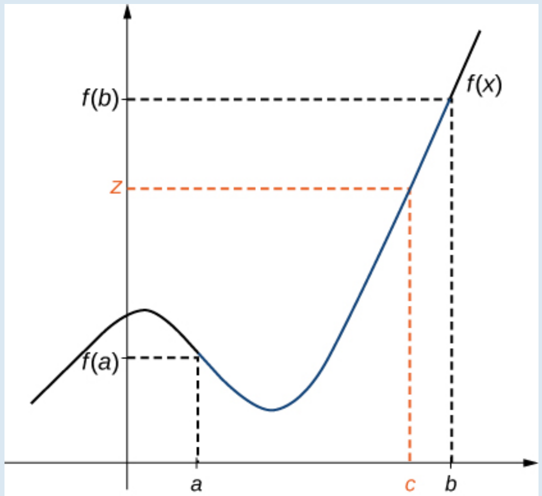


Figure 2.38 There is a number $c \in [a, b]$ that satisfies $f(c) = z$.

Example 2.36

Application of the Intermediate Value Theorem

Show that $f(x) = x - \cos x$ has at least one zero.

Solution

Since $f(x) = x - \cos x$ is continuous over $(-\infty, +\infty)$, it is continuous over any closed interval of the form $[a, b]$. If you can find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs, you can use the Intermediate Value Theorem to conclude there must be a real number c in (a, b) that satisfies $f(c) = 0$. Note that

$$f(0) = 0 - \cos(0) = -1 < 0$$

and

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2} > 0.$$

Using the Intermediate Value Theorem, we can see that there must be a real number c in $[0, \pi/2]$ that satisfies $f(c) = 0$. Therefore, $f(x) = x - \cos x$ has at least one zero.

Example 2.37

When Can You Apply the Intermediate Value Theorem?

If $f(x)$ is continuous over $[0, 2]$, $f(0) > 0$ and $f(2) > 0$, can we use the Intermediate Value Theorem to conclude that $f(x)$ has no zeros in the interval $[0, 2]$? Explain.

Solution

No. The Intermediate Value Theorem only allows us to conclude that we can find a value between $f(0)$ and $f(2)$; it doesn't allow us to conclude that we can't find other values. To see this more clearly, consider the function $f(x) = (x-1)^2$. It satisfies $f(0) = 1 > 0, f(2) = 1 > 0,$ and $f(1) = 0$.

Example 2.38

When Can You Apply the Intermediate Value Theorem?

For $f(x) = 1/x, f(-1) = -1 < 0$ and $f(1) = 1 > 0$. Can we conclude that $f(x)$ has a zero in the interval $[-1, 1]$?

Solution

No. The function is not continuous over $[-1, 1]$. The Intermediate Value Theorem does not apply here.

2.26 Show that $f(x) = x^3 - x^2 - 3x + 1$ has a zero over the interval $[0, 1]$.

2.4 EXERCISES

For the following exercises, determine the point(s), if any, at which each function is discontinuous. Classify any discontinuity as jump, removable, infinite, or other.

131. $f(x) = \frac{1}{\sqrt{x}}$

132. $f(x) = \frac{2}{x^2 + 1}$

133. $f(x) = \frac{x}{x^2 - x}$

134. $g(t) = t^{-1} + 1$

135. $f(x) = \frac{5}{e^x - 2}$

136. $f(x) = \frac{|x - 2|}{x - 2}$

137. $H(x) = \tan 2x$

138. $f(t) = \frac{t + 3}{t^2 + 5t + 6}$

For the following exercises, decide if the function continuous at the given point. If it is discontinuous, what type of discontinuity is it?

139. $f(x)\frac{2x^2 - 5x + 3}{x - 1}$ at $x = 1$

140. $h(\theta) = \frac{\sin\theta - \cos\theta}{\tan\theta}$ at $\theta = \pi$

141. $g(u) = \begin{cases} \frac{6u^2 + u - 2}{2u - 1} & \text{if } u \neq \frac{1}{2} \\ \frac{7}{2} & \text{if } u = \frac{1}{2} \end{cases}$, at $u = \frac{1}{2}$

142. $f(y) = \frac{\sin(\pi y)}{\tan(\pi y)}$, at $y = 1$

143. $f(x) = \begin{cases} x^2 - e^x & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases}$, at $x = 0$

144. $f(x) = \begin{cases} x\sin(x) \text{ if } x \leq \pi \\ x\tan(x) \text{ if } x > \pi \end{cases}$, at $x = \pi$

In the following exercises, find the value(s) of k that makes each function continuous over the given interval.

145. $f(x) = \begin{cases} 3x + 2, & x < k \\ 2x - 3, & k \leq x \leq 8 \end{cases}$

146. $f(\theta) = \begin{cases} \sin\theta, & 0 \leq \theta < \frac{\pi}{2} \\ \cos(\theta + k), & \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$

147. $f(x) = \begin{cases} \frac{x^2 + 3x + 2}{x + 2}, & x \neq -2 \\ k, & x = -2 \end{cases}$

148. $f(x) = \begin{cases} e^{kx}, & 0 \leq x < 4 \\ x + 3, & 4 \leq x \leq 8 \end{cases}$

149. $f(x) = \begin{cases} \sqrt{kx}, & 0 \leq x \leq 3 \\ x + 1, & 3 < x \leq 10 \end{cases}$

In the following exercises, use the Intermediate Value Theorem (IVT).

150. Let $h(x) = \begin{cases} 3x^2 - 4, & x \leq 2 \\ 5 + 4x, & x > 2 \end{cases}$ Over the interval $[0, 4]$, there is no value of x such that $h(x) = 10$, although $h(0) < 10$ and $h(4) > 10$. Explain why this does not contradict the IVT.

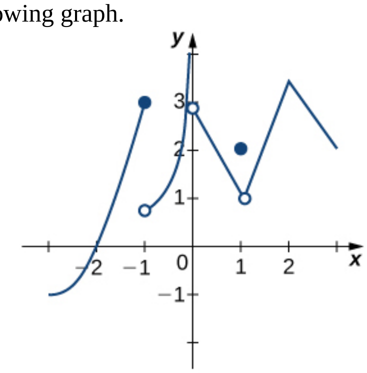
151. A particle moving along a line has at each time t a position function $s(t)$, which is continuous. Assume $s(2) = 5$ and $s(5) = 2$. Another particle moves such that its position is given by $h(t) = s(t) - t$. Explain why there must be a value c for $2 < c < 5$ such that $h(c) = 0$.

152. **[T]** Use the statement "The cosine of t is equal to t cubed."

a. Write a mathematical equation of the statement.
b. Prove that the equation in part a. has at least one real solution.
c. Use a calculator to find an interval of length 0.01 that contains a solution.

153. Apply the IVT to determine whether $2^x = x^3$ has a solution in one of the intervals $[1.25, 1.375]$ or $[1.375, 1.5]$. Briefly explain your response for each interval.

154. Consider the graph of the function $y = f(x)$ shown in the following graph.



a. Find all values for which the function is discontinuous.
b. For each value in part a., state why the formal definition of continuity does not apply.
c. Classify each discontinuity as either jump, removable, or infinite.

155. Let $f(x) = \begin{cases} 3x, & x > 1 \\ x^3, & x < 1 \end{cases}$.

a. Sketch the graph of *f*.
b. Is it possible to find a value *k* such that $f(1) = k$, which makes $f(x)$ continuous for all real numbers? Briefly explain.

156. Let $f(x) = \dfrac{x^4 - 1}{x^2 - 1}$ for $x \neq -1, 1$.

a. Sketch the graph of *f*.
b. Is it possible to find values k_1 and k_2 such that $f(-1) = k_1$ and $f(1) = k_2$, and that makes $f(x)$ continuous for all real numbers? Briefly explain.

157. Sketch the graph of the function $y = f(x)$ with properties i. through vii.

i. The domain of *f* is $(-\infty, +\infty)$.
ii. *f* has an infinite discontinuity at $x = -6$.
iii. $f(-6) = 3$
iv. $\lim_{x \to -3^-} f(x) = \lim_{x \to -3^+} f(x) = 2$
v. $f(-3) = 3$
vi. *f* is left continuous but not right continuous at $x = 3$.
vii. $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$

158. Sketch the graph of the function $y = f(x)$ with properties i. through iv.

i. The domain of *f* is $[0, 5]$.
ii. $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$ exist and are equal.
iii. $f(x)$ is left continuous but not continuous at $x = 2$, and right continuous but not continuous at $x = 3$.
iv. $f(x)$ has a removable discontinuity at $x = 1$, a jump discontinuity at $x = 2$, and the following limits hold: $\lim_{x \to 3^-} f(x) = -\infty$ and $\lim_{x \to 3^+} f(x) = 2$.

In the following exercises, suppose $y = f(x)$ is defined for all *x*. For each description, sketch a graph with the indicated property.

159. Discontinuous at $x = 1$ with $\lim_{x \to -1} f(x) = -1$ and $\lim_{x \to 2} f(x) = 4$

160. Discontinuous at $x = 2$ but continuous elsewhere with $\lim_{x \to 0} f(x) = \frac{1}{2}$

Determine whether each of the given statements is true. Justify your response with an explanation or counterexample.

161. $f(t) = \dfrac{2}{e^t - e^{-t}}$ is continuous everywhere.

162. If the left- and right-hand limits of $f(x)$ as $x \to a$ exist and are equal, then *f* cannot be discontinuous at $x = a$.

163. If a function is not continuous at a point, then it is not defined at that point.

164. According to the IVT, $\cos x - \sin x - x = 2$ has a solution over the interval $[-1, 1]$.

165. If $f(x)$ is continuous such that $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ has exactly one solution in $[a, b]$.

166. The function $f(x) = \dfrac{x^2 - 4x + 3}{x^2 - 1}$ is continuous over the interval $[0, 3]$.

167. If $f(x)$ is continuous everywhere and $f(a),\ f(b) > 0,$ then there is no root of $f(x)$ in the interval $[a,\ b]$.

[T] The following problems consider the scalar form of Coulomb's law, which describes the electrostatic force between two point charges, such as electrons. It is given by the equation $F(r) = k_e \frac{|q_1 q_2|}{r^2},$ where k_e is Coulomb's constant, q_i are the magnitudes of the charges of the two particles, and r is the distance between the two particles.

168. To simplify the calculation of a model with many interacting particles, after some threshold value $r = R,$ we approximate F as zero.

a. Explain the physical reasoning behind this assumption.
b. What is the force equation?
c. Evaluate the force F using both Coulomb's law and our approximation, assuming two protons with a charge magnitude of 1.6022×10^{-19} coulombs (C), and the Coulomb constant $k_e = 8.988 \times 10^9\,\text{Nm}^2/\text{C}^2$ are 1 m apart. Also, assume $R < 1\,\text{m}.$ How much inaccuracy does our approximation generate? Is our approximation reasonable?
d. Is there any finite value of R for which this system remains continuous at R?

169. Instead of making the force 0 at R, instead we let the force be 10^{-20} for $r \geq R.$ Assume two protons, which have a magnitude of charge 1.6022×10^{-19} C, and the Coulomb constant $k_e = 8.988 \times 10^9\,\text{Nm}^2/\text{C}^2.$ Is there a value R that can make this system continuous? If so, find it.

Recall the discussion on spacecraft from the chapter opener. The following problems consider a rocket launch from Earth's surface. The force of gravity on the rocket is given by $F(d) = -mk/d^2,$ where m is the mass of the rocket, d is the distance of the rocket from the center of Earth, and k is a constant.

170. **[T]** Determine the value and units of k given that the mass of the rocket is 3 million kg. (*Hint*: The distance from the center of Earth to its surface is 6378 km.)

171. **[T]** After a certain distance D has passed, the gravitational effect of Earth becomes quite negligible, so we can approximate the force function by $F(d) = \begin{cases} -\frac{mk}{d^2} & \text{if } d < D \\ 10{,}000 & \text{if } d \geq D \end{cases}.$ Using the value of k found in the previous exercise, find the necessary condition D such that the force function remains continuous.

172. As the rocket travels away from Earth's surface, there is a distance D where the rocket sheds some of its mass, since it no longer needs the excess fuel storage. We can write this function as $F(d) = \begin{cases} -\frac{m_1 k}{d^2} & \text{if } d < D \\ -\frac{m_2 k}{d^2} & \text{if } d \geq D \end{cases}.$ Is there a D value such that this function is continuous, assuming $m_1 \neq m_2$?

Prove the following functions are continuous everywhere

173. $f(\theta) = \sin\theta$

174. $g(x) = |x|$

175. Where is $f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$ continuous?

2.5 | The Precise Definition of a Limit

Learning Objectives

2.5.1 Describe the epsilon-delta definition of a limit.
2.5.2 Apply the epsilon-delta definition to find the limit of a function.
2.5.3 Describe the epsilon-delta definitions of one-sided limits and infinite limits.
2.5.4 Use the epsilon-delta definition to prove the limit laws.

By now you have progressed from the very informal definition of a limit in the introduction of this chapter to the intuitive understanding of a limit. At this point, you should have a very strong intuitive sense of what the limit of a function means and how you can find it. In this section, we convert this intuitive idea of a limit into a formal definition using precise mathematical language. The formal definition of a limit is quite possibly one of the most challenging definitions you will encounter early in your study of calculus; however, it is well worth any effort you make to reconcile it with your intuitive notion of a limit. Understanding this definition is the key that opens the door to a better understanding of calculus.

Quantifying Closeness

Before stating the formal definition of a limit, we must introduce a few preliminary ideas. Recall that the distance between two points a and b on a number line is given by $|a-b|$.

- The statement $|f(x)-L|<\varepsilon$ may be interpreted as: *The distance between* $f(x)$ *and L is less than ε.*
- The statement $0<|x-a|<\delta$ may be interpreted as: $x \neq a$ *and the distance between x and a is less than δ.*

It is also important to look at the following equivalences for absolute value:

- The statement $|f(x)-L|<\varepsilon$ is equivalent to the statement $L-\varepsilon<f(x)<L+\varepsilon$.
- The statement $0<|x-a|<\delta$ is equivalent to the statement $a-\delta<x<a+\delta$ and $x \neq a$.

With these clarifications, we can state the formal **epsilon-delta definition of the limit**.

Definition

Let $f(x)$ be defined for all $x \neq a$ over an open interval containing a. Let L be a real number. Then

$$\lim_{x \to a} f(x) = L$$

if, for every $\varepsilon>0$, there exists a $\delta>0$, such that if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$.

This definition may seem rather complex from a mathematical point of view, but it becomes easier to understand if we break it down phrase by phrase. The statement itself involves something called a *universal quantifier* (for every $\varepsilon>0$), an *existential quantifier* (there exists a $\delta>0$), and, last, a *conditional statement* (if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$). Let's take a look at **Table 2.9**, which breaks down the definition and translates each part.

Definition	Translation
1. For every $\varepsilon > 0,$	1. For every positive distance ε from L,
2. there exists a $\delta > 0,$	2. There is a positive distance δ from a,
3. such that	3. such that
4. if $0 < \lvert x - a \rvert < \delta,$ then $\lvert f(x) - L \rvert < \varepsilon.$	4. if x is closer than δ to a and $x \neq a,$ then $f(x)$ is closer than ε to L.

Table 2.9 Translation of the Epsilon-Delta Definition of the Limit

We can get a better handle on this definition by looking at the definition geometrically. **Figure 2.39** shows possible values of δ for various choices of $\varepsilon > 0$ for a given function $f(x),$ a number a, and a limit L at a. Notice that as we choose smaller values of ε (the distance between the function and the limit), we can always find a δ small enough so that if we have chosen an x value within δ of a, then the value of $f(x)$ is within ε of the limit L.

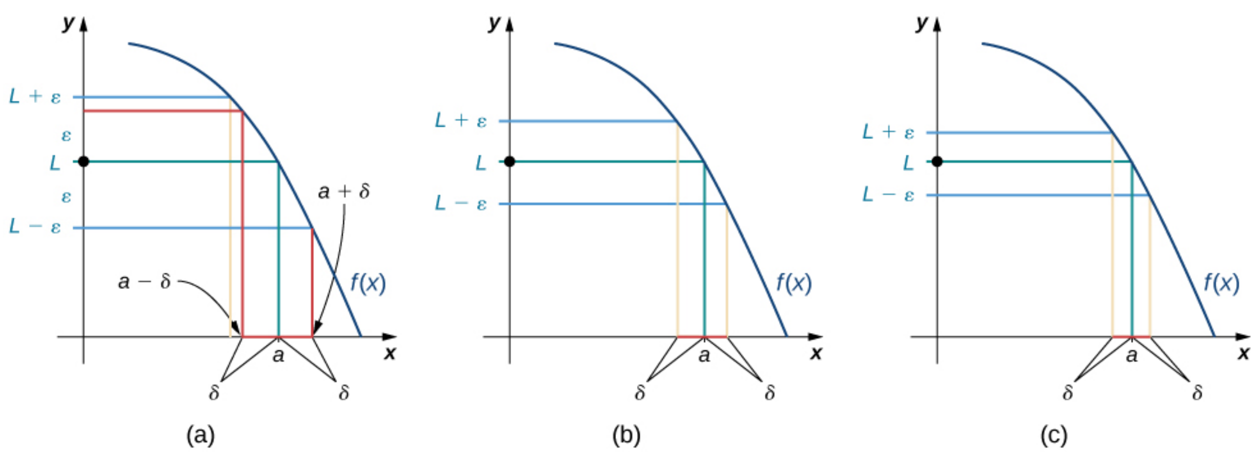


Figure 2.39 These graphs show possible values of δ, given successively smaller choices of ε.

Visit the following applet to experiment with finding values of δ for selected values of ε:

- **The epsilon-delta definition of limit (http://www.openstax.org/l/20_epsilondelt)**

Example 2.39 shows how you can use this definition to prove a statement about the limit of a specific function at a specified value.

Example 2.39

Proving a Statement about the Limit of a Specific Function

Prove that $\lim_{x \to 1} (2x + 1) = 3.$

Solution

Let $\varepsilon > 0$.

The first part of the definition begins "For every $\varepsilon > 0$." This means we must prove that whatever follows is true no matter what positive value of ε is chosen. By stating "Let $\varepsilon > 0$," we signal our intent to do so.

Choose $\delta = \frac{\varepsilon}{2}$.

The definition continues with "there exists a $\delta > 0$." The phrase "there exists" in a mathematical statement is always a signal for a scavenger hunt. In other words, we must go and find δ. So, where exactly did $\delta = \varepsilon/2$ come from? There are two basic approaches to tracking down δ. One method is purely algebraic and the other is geometric.

We begin by tackling the problem from an algebraic point of view. Since ultimately we want $|(2x+1)-3| < \varepsilon$, we begin by manipulating this expression: $|(2x+1)-3| < \varepsilon$ is equivalent to $|2x-2| < \varepsilon$, which in turn is equivalent to $|2||x-1| < \varepsilon$. Last, this is equivalent to $|x-1| < \varepsilon/2$. Thus, it would seem that $\delta = \varepsilon/2$ is appropriate.

We may also find δ through geometric methods. **Figure 2.40** demonstrates how this is done.

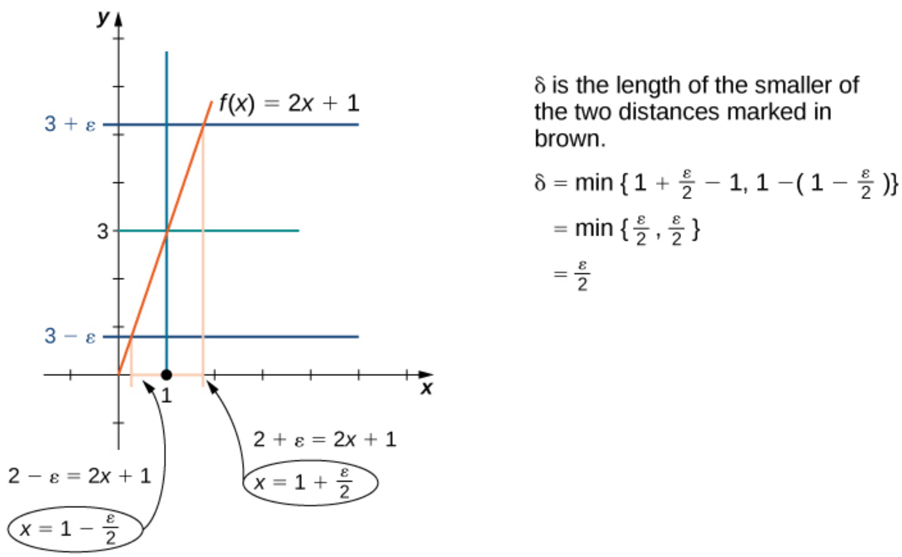


Figure 2.40 This graph shows how we find δ geometrically.

Assume $0 < |x-1| < \delta$. When δ has been chosen, our goal is to show that if $0 < |x-1| < \delta$, then $|(2x+1)-3| < \varepsilon$. To prove any statement of the form "If this, then that," we begin by assuming "this" and trying to get "that."

Thus,

$$\begin{aligned}
|(2x+1)-3| &= |2x-2| && \text{property of absolute value} \\
&= |2(x-1)| && \\
&= |2||x-1| && |2| = 2 \\
&= 2|x-1| && \\
&< 2\cdot\delta && \text{here's where we use the assumption that } 0 < |x-1| < \delta \\
&= 2\cdot\frac{\varepsilon}{2} = \varepsilon && \text{here's where we use our choice of } \delta = \varepsilon/2
\end{aligned}$$

Analysis

In this part of the proof, we started with $|(2x+1)-3|$ and used our assumption $0 < |x-1| < \delta$ in a key part of the chain of inequalities to get $|(2x+1)-3|$ to be less than ε. We could just as easily have manipulated the assumed inequality $0 < |x-1| < \delta$ to arrive at $|(2x+1)-3| < \varepsilon$ as follows:

$$\begin{aligned} 0 < |x-1| < \delta &\Rightarrow |x-1| < \delta \\ &\Rightarrow -\delta < x-1 < \delta \\ &\Rightarrow -\frac{\varepsilon}{2} < x-1 < \frac{\varepsilon}{2} \\ &\Rightarrow -\varepsilon < 2x-2 < \varepsilon \\ &\Rightarrow -\varepsilon < 2x-2 < \varepsilon \\ &\Rightarrow |2x-2| < \varepsilon \\ &\Rightarrow |(2x+1)-3| < \varepsilon. \end{aligned}$$

Therefore, $\lim_{x \to 1}(2x+1) = 3.$ (Having completed the proof, we state what we have accomplished.)

After removing all the remarks, here is a final version of the proof:

Let $\varepsilon > 0.$

Choose $\delta = \varepsilon/2.$

Assume $0 < |x-1| < \delta.$

Thus,

$$\begin{aligned} |(2x+1)-3| &= |2x-2| \\ &= |2(x-1)| \\ &= |2||x-1| \\ &= 2|x-1| \\ &< 2 \cdot \delta \\ &= 2 \cdot \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Therefore, $\lim_{x \to 1}(2x+1) = 3.$

The following Problem-Solving Strategy summarizes the type of proof we worked out in **Example 2.39**.

Problem-Solving Strategy: Proving That $\lim_{x \to a} f(x) = L$ for a Specific Function $f(x)$

1. Let's begin the proof with the following statement: Let $\varepsilon > 0.$
2. Next, we need to obtain a value for $\delta.$ After we have obtained this value, we make the following statement, filling in the blank with our choice of δ: Choose $\delta =$ ________.
3. The next statement in the proof should be (at this point, we fill in our given value for a):
Assume $0 < |x-a| < \delta.$
4. Next, based on this assumption, we need to show that $|f(x)-L| < \varepsilon,$ where $f(x)$ and L are our function $f(x)$ and our limit L. At some point, we need to use $0 < |x-a| < \delta.$
5. We conclude our proof with the statement: Therefore, $\lim_{x \to a} f(x) = L.$

Example 2.40

Proving a Statement about a Limit

Complete the proof that $\lim\limits_{x \to -1}(4x + 1) = -3$ by filling in the blanks.

Let ______.

Choose $\delta = \underline{\qquad\quad}$.

Assume $0 < |x - \underline{\qquad\quad}| < \delta$.

Thus, $|\underline{\qquad\quad} - \underline{\qquad\quad}| = \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}\varepsilon$.

Solution

We begin by filling in the blanks where the choices are specified by the definition. Thus, we have

Let $\varepsilon > 0$.

Choose $\delta = \underline{\qquad\quad}$.

Assume $0 < |x - (-1)| < \delta$. (or equivalently, $0 < |x + 1| < \delta$.)

Thus, $|(4x + 1) - (-3)| = |4x + 4| = |4||x + 1| < 4\delta\underline{\qquad\quad}\varepsilon$.

Focusing on the final line of the proof, we see that we should choose $\delta = \frac{\varepsilon}{4}$.

We now complete the final write-up of the proof:

Let $\varepsilon > 0$.

Choose $\delta = \frac{\varepsilon}{4}$.

Assume $0 < |x - (-1)| < \delta$ (or equivalently, $0 < |x + 1| < \delta$.)

Thus, $|(4x + 1) - (-3)| = |4x + 4| = |4||x + 1| < 4\delta = 4(\varepsilon/4) = \varepsilon$.

2.27 Complete the proof that $\lim\limits_{x \to 2}(3x - 2) = 4$ by filling in the blanks.

Let ________.

Choose $\delta = \underline{\qquad\quad}$.

Assume $0 < |x - \underline{\qquad}| < \underline{\qquad}$.

Thus,

$|\underline{\qquad\quad} - \underline{\qquad}| = \underline{\qquad\qquad\qquad\qquad\qquad\qquad}\varepsilon$.

Therefore, $\lim\limits_{x \to 2}(3x - 2) = 4$.

In **Example 2.39** and **Example 2.40**, the proofs were fairly straightforward, since the functions with which we were working were linear. In **Example 2.41**, we see how to modify the proof to accommodate a nonlinear function.

Example 2.41

Proving a Statement about the Limit of a Specific Function (Geometric Approach)

Prove that $\lim_{x \to 2} x^2 = 4.$

Solution

1. Let $\varepsilon > 0.$ The first part of the definition begins "For every $\varepsilon > 0,$" so we must prove that whatever follows is true no matter what positive value of ε is chosen. By stating "Let $\varepsilon > 0,$" we signal our intent to do so.
2. Without loss of generality, assume $\varepsilon \le 4.$ Two questions present themselves: Why do we want $\varepsilon \le 4$ and why is it okay to make this assumption? In answer to the first question: Later on, in the process of solving for $\delta,$ we will discover that δ involves the quantity $\sqrt{4-\varepsilon}.$ Consequently, we need $\varepsilon \le 4.$ In answer to the second question: If we can find $\delta > 0$ that "works" for $\varepsilon \le 4,$ then it will "work" for any $\varepsilon > 4$ as well. Keep in mind that, although it is always okay to put an upper bound on ε, it is never okay to put a lower bound (other than zero) on ε.
3. Choose $\delta = \min\{2 - \sqrt{4-\varepsilon}, \sqrt{4+\varepsilon} - 2\}.$ **Figure 2.41** shows how we made this choice of $\delta.$

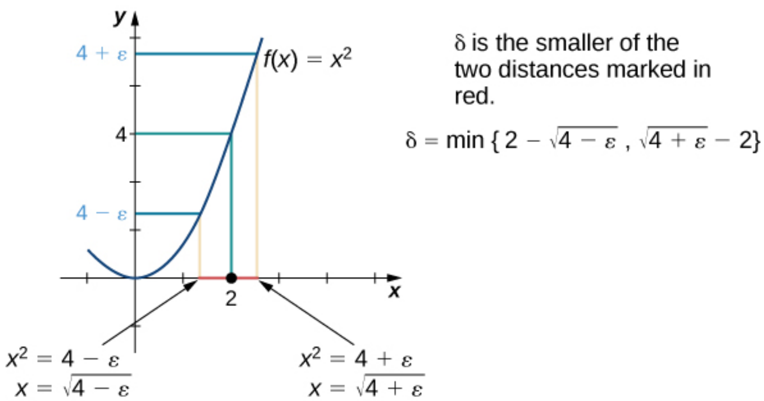


Figure 2.41 This graph shows how we find δ geometrically for a given ε for the proof in **Example 2.41**.

4. We must show: If $0 < |x-2| < \delta,$ then $|x^2 - 4| < \varepsilon,$ so we must begin by assuming

$$0 < |x-2| < \delta.$$

We don't really need $0 < |x-2|$ (in other words, $x \ne 2$) for this proof. Since $0 < |x-2| < \delta \Rightarrow |x-2| < \delta,$ it is okay to drop $0 < |x-2|.$

$$|x-2| < \delta.$$

Hence,

$$-\delta < x - 2 < \delta.$$

Recall that $\delta = \min\{2 - \sqrt{4-\varepsilon}, \sqrt{4+\varepsilon} - 2\}.$ Thus, $\delta \le 2 - \sqrt{4-\varepsilon}$ and consequently $-\left(2 - \sqrt{4-\varepsilon}\right) \le -\delta.$ We also use $\delta \le \sqrt{4+\varepsilon} - 2$ here. We might ask at this point: Why did we substitute $2 - \sqrt{4-\varepsilon}$ for δ on the left-hand side of the inequality and $\sqrt{4+\varepsilon} - 2$ on the right-hand side of the inequality? If we look at **Figure 2.41**, we see that $2 - \sqrt{4-\varepsilon}$ corresponds to the distance on the left of 2 on the x-axis and $\sqrt{4+\varepsilon}-2$ corresponds to the distance on the right. Thus,

$$-\left(2-\sqrt{4-\varepsilon}\right) \le -\delta < x-2 < \delta \le \sqrt{4+\varepsilon}-2.$$

We simplify the expression on the left:

$$-2+\sqrt{4-\varepsilon} < x-2 < \sqrt{4+\varepsilon}-2.$$

Then, we add 2 to all parts of the inequality:

$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}.$$

We square all parts of the inequality. It is okay to do so, since all parts of the inequality are positive:

$$4-\varepsilon < x^2 < 4+\varepsilon.$$

We subtract 4 from all parts of the inequality:

$$-\varepsilon < x^2-4 < \varepsilon.$$

Last,

$$|x^2-4| < \varepsilon.$$

5. Therefore,

$$\lim_{x\to 2} x^2 = 4.$$

2.28 Find δ corresponding to $\varepsilon > 0$ for a proof that $\lim_{x\to 9}\sqrt{x} = 3.$

The geometric approach to proving that the limit of a function takes on a specific value works quite well for some functions. Also, the insight into the formal definition of the limit that this method provides is invaluable. However, we may also approach limit proofs from a purely algebraic point of view. In many cases, an algebraic approach may not only provide us with additional insight into the definition, it may prove to be simpler as well. Furthermore, an algebraic approach is the primary tool used in proofs of statements about limits. For **Example 2.42**, we take on a purely algebraic approach.

Example 2.42

Proving a Statement about the Limit of a Specific Function (Algebraic Approach)

Prove that $\lim_{x\to -1}\left(x^2-2x+3\right) = 6.$

Solution

Let's use our outline from the Problem-Solving Strategy:

1. Let $\varepsilon > 0$.
2. Choose $\delta = \min\{1, \varepsilon/5\}$. This choice of δ may appear odd at first glance, but it was obtained by

taking a look at our ultimate desired inequality: $\left|\left(x^2 - 2x + 3\right) - 6\right| < \varepsilon$. This inequality is equivalent to $|x + 1| \cdot |x - 3| < \varepsilon$. At this point, the temptation simply to choose $\delta = \dfrac{\varepsilon}{x - 3}$ is very strong. Unfortunately, our choice of δ must depend on ε only and no other variable. If we can replace $|x - 3|$ by a numerical value, our problem can be resolved. This is the place where assuming $\delta \leq 1$ comes into play. The choice of $\delta \leq 1$ here is arbitrary. We could have just as easily used any other positive number. In some proofs, greater care in this choice may be necessary. Now, since $\delta \leq 1$ and $|x + 1| < \delta \leq 1$, we are able to show that $|x - 3| < 5$. Consequently, $|x + 1| \cdot |x - 3| < |x + 1| \cdot 5$. At this point we realize that we also need $\delta \leq \varepsilon/5$. Thus, we choose $\delta = \min\{1, \varepsilon/5\}$.

3. Assume $0 < |x + 1| < \delta$. Thus,

$$|x + 1| < 1 \text{ and } |x + 1| < \frac{\varepsilon}{5}.$$

Since $|x + 1| < 1$, we may conclude that $-1 < x + 1 < 1$. Thus, by subtracting 4 from all parts of the inequality, we obtain $-5 < x - 3 < -1$. Consequently, $|x - 3| < 5$. This gives us

$$\left|\left(x^2 - 2x + 3\right) - 6\right| = |x + 1| \cdot |x - 3| < \frac{\varepsilon}{5} \cdot 5 = \varepsilon.$$

Therefore,

$$\lim_{x \to -1} \left(x^2 - 2x + 3\right) = 6.$$

2.29 Complete the proof that $\lim\limits_{x \to 1} x^2 = 1$.

Let $\varepsilon > 0$; choose $\delta = \min\{1, \varepsilon/3\}$; assume $0 < |x - 1| < \delta$.

Since $|x - 1| < 1$, we may conclude that $-1 < x - 1 < 1$. Thus, $1 < x + 1 < 3$. Hence, $|x + 1| < 3$.

You will find that, in general, the more complex a function, the more likely it is that the algebraic approach is the easiest to apply. The algebraic approach is also more useful in proving statements about limits.

Proving Limit Laws

We now demonstrate how to use the epsilon-delta definition of a limit to construct a rigorous proof of one of the limit laws. The **triangle inequality** is used at a key point of the proof, so we first review this key property of absolute value.

Definition

The **triangle inequality** states that if a and b are any real numbers, then $|a + b| \leq |a| + |b|$.

Proof

We prove the following limit law: If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, then $\lim\limits_{x \to a} (f(x) + g(x)) = L + M$.

Let $\varepsilon > 0$.

Choose $\delta_1 > 0$ so that if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \varepsilon/2$.

Choose $\delta_2 > 0$ so that if $0 < |x - a| < \delta_2$, then $|g(x) - M| < \varepsilon/2$.

Choose $\delta = \min\{\delta_1, \delta_2\}$.

Assume $0 < |x - a| < \delta$.

Thus,

$$0 < |x - a| < \delta_1 \text{ and } 0 < |x - a| < \delta_2.$$

Hence,

$$\begin{aligned}|(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\le |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

□

We now explore what it means for a limit not to exist. The limit $\lim_{x \to a} f(x)$ does not exist if there is no real number L for which $\lim_{x \to a} f(x) = L$. Thus, for all real numbers L, $\lim_{x \to a} f(x) \ne L$. To understand what this means, we look at each part of the definition of $\lim_{x \to a} f(x) = L$ together with its opposite. A translation of the definition is given in **Table 2.10**.

Definition	**Opposite**
1. For every $\varepsilon > 0$,	1. There exists $\varepsilon > 0$ so that
2. there exists a $\delta > 0$, so that	2. for every $\delta > 0$,
3. if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.	3. There is an x satisfying $0 < |x - a| < \delta$ so that $|f(x) - L| \ge \varepsilon$.

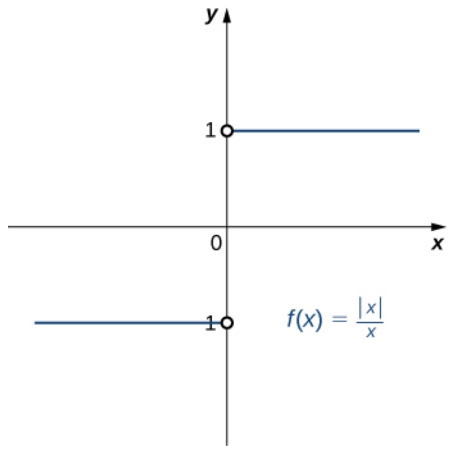
Table 2.10 Translation of the Definition of $\lim_{x \to a} f(x) = L$ and its Opposite

Finally, we may state what it means for a limit not to exist. The limit $\lim_{x \to a} f(x)$ does not exist if for every real number L, there exists a real number $\varepsilon > 0$ so that for all $\delta > 0$, there is an x satisfying $0 < |x - a| < \delta$, so that $|f(x) - L| \ge \varepsilon$. Let's apply this in **Example 2.43** to show that a limit does not exist.

Example 2.43

Showing That a Limit Does Not Exist

Show that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist. The graph of $f(x) = |x|/x$ is shown here:



Solution

Suppose that L is a candidate for a limit. Choose $\varepsilon = 1/2$.

Let $\delta > 0$. Either $L \geq 0$ or $L < 0$. If $L \geq 0$, then let $x = -\delta/2$. Thus,

$$|x - 0| = \left|-\frac{\delta}{2} - 0\right| = \frac{\delta}{2} < \delta$$

and

$$\left|\frac{\left|-\frac{\delta}{2}\right|}{-\frac{\delta}{2}} - L\right| = |-1 - L| = L + 1 \geq 1 > \frac{1}{2} = \varepsilon.$$

On the other hand, if $L < 0$, then let $x = \delta/2$. Thus,

$$|x - 0| = \left|\frac{\delta}{2} - 0\right| = \frac{\delta}{2} < \delta$$

and

$$\left|\frac{\left|\frac{\delta}{2}\right|}{\frac{\delta}{2}} - L\right| = |1 - L| = |L| + 1 \geq 1 > \frac{1}{2} = \varepsilon.$$

Thus, for any value of L, $\lim\limits_{x \to 0} \frac{|x|}{x} \neq L$.

One-Sided and Infinite Limits

Just as we first gained an intuitive understanding of limits and then moved on to a more rigorous definition of a limit, we now revisit one-sided limits. To do this, we modify the epsilon-delta definition of a limit to give formal epsilon-delta definitions for limits from the right and left at a point. These definitions only require slight modifications from the definition of the limit. In the definition of the limit from the right, the inequality $0 < x - a < \delta$ replaces $0 < |x - a| < \delta$, which ensures that we only consider values of x that are greater than (to the right of) a. Similarly, in the definition of the limit from the left, the inequality $-\delta < x - a < 0$ replaces $0 < |x - a| < \delta$, which ensures that we only consider values of x that are less than (to the left of) a.

Definition

Limit from the Right: Let $f(x)$ be defined over an open interval of the form (a, b) where $a < b$. Then,

$$\lim_{x \to a^+} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < x - a < \delta$, then $|f(x) - L| < \varepsilon$.

Limit from the Left: Let $f(x)$ be defined over an open interval of the form (b, c) where $b < c$. Then,

$$\lim_{x \to a^-} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $-\delta < x - a < 0$, then $|f(x) - L| < \varepsilon$.

Example 2.44

Proving a Statement about a Limit From the Right

Prove that $\lim_{x \to 4^+} \sqrt{x - 4} = 0$.

Solution

Let $\varepsilon > 0$.

Choose $\delta = \varepsilon^2$. Since we ultimately want $\left|\sqrt{x - 4} - 0\right| < \varepsilon$, we manipulate this inequality to get $\sqrt{x - 4} < \varepsilon$ or, equivalently, $0 < x - 4 < \varepsilon^2$, making $\delta = \varepsilon^2$ a clear choice. We may also determine δ geometrically, as shown in Figure 2.42.

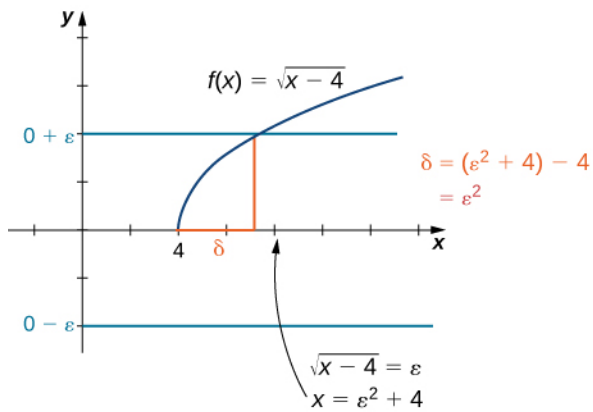


Figure 2.42 This graph shows how we find δ for the proof in Example 2.44.

Assume $0 < x - 4 < \delta$. Thus, $0 < x - 4 < \varepsilon^2$. Hence, $0 < \sqrt{x - 4} < \varepsilon$. Finally, $\left|\sqrt{x - 4} - 0\right| < \varepsilon$.

Therefore, $\lim_{x \to 4^+} \sqrt{x - 4} = 0$.

2.30 Find δ corresponding to ε for a proof that $\lim_{x \to 1^-} \sqrt{1 - x} = 0$.

We conclude the process of converting our intuitive ideas of various types of limits to rigorous formal definitions by pursuing a formal definition of infinite limits. To have $\lim_{x \to a} f(x) = +\infty,$ we want the values of the function $f(x)$ to get larger and larger as x approaches a. Instead of the requirement that $|f(x) - L| < \varepsilon$ for arbitrarily small ε when $0 < |x - a| < \delta$ for small enough $\delta,$ we want $f(x) > M$ for arbitrarily large positive M when $0 < |x - a| < \delta$ for small enough δ. **Figure 2.43** illustrates this idea by showing the value of δ for successively larger values of M.

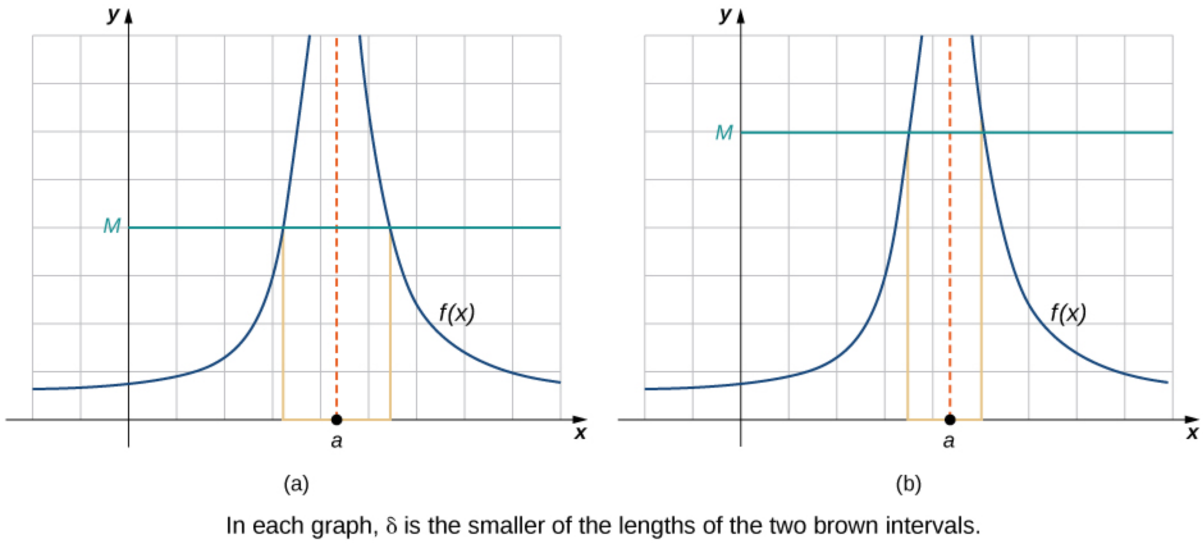


Figure 2.43 These graphs plot values of δ for M to show that $\lim_{x \to a} f(x) = +\infty$.

Definition

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Then, we have an infinite limit

$$\lim_{x \to a} f(x) = +\infty$$

if for every $M > 0,$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta,$ then $f(x) > M$.

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Then, we have a negative infinite limit

$$\lim_{x \to a} f(x) = -\infty$$

if for every $M > 0,$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta,$ then $f(x) < -M$.

2.5 EXERCISES

In the following exercises, write the appropriate $\varepsilon - \delta$ definition for each of the given statements.

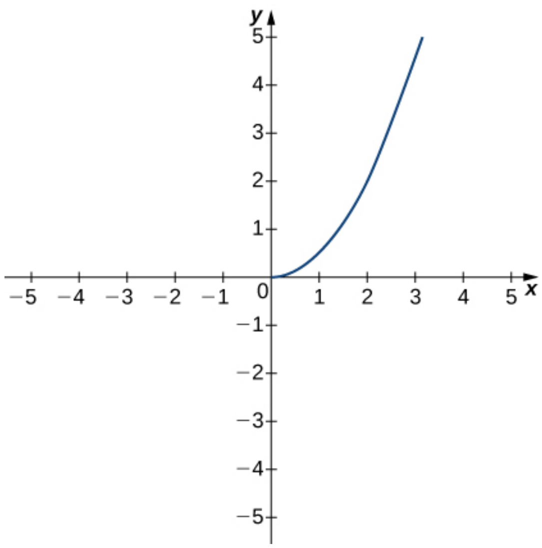
176. $\lim_{x \to a} f(x) = N$

177. $\lim_{t \to b} g(t) = M$

178. $\lim_{x \to c} h(x) = L$

179. $\lim_{x \to a} \varphi(x) = A$

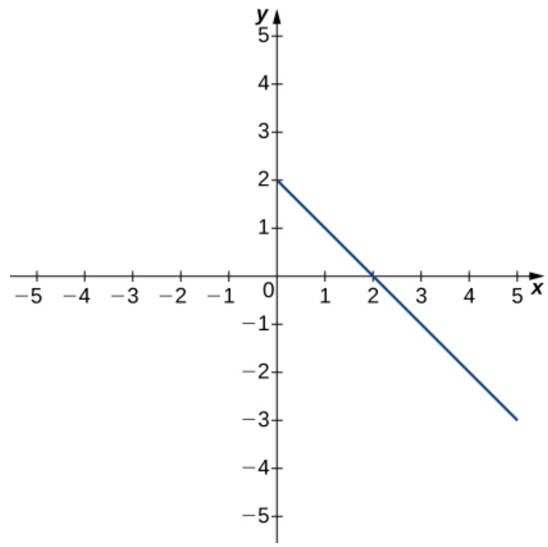
The following graph of the function f satisfies $\lim_{x \to 2} f(x) = 2$. In the following exercises, determine a value of $\delta > 0$ that satisfies each statement.



180. If $0 < |x - 2| < \delta$, then $|f(x) - 2| < 1$.

181. If $0 < |x - 2| < \delta$, then $|f(x) - 2| < 0.5$.

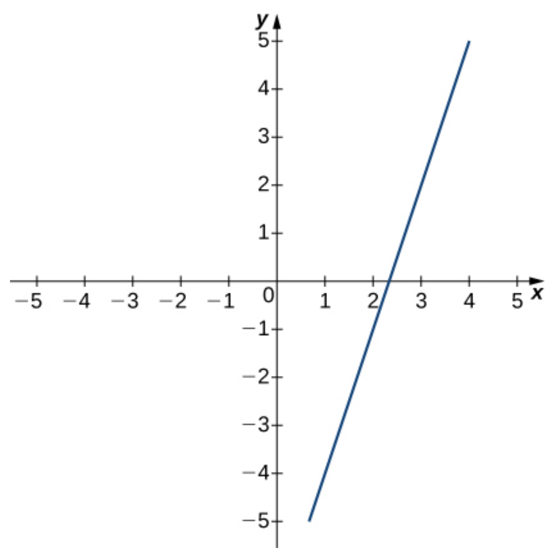
The following graph of the function f satisfies $\lim_{x \to 3} f(x) = -1$. In the following exercises, determine a value of $\delta > 0$ that satisfies each statement.



182. If $0 < |x - 3| < \delta$, then $|f(x) + 1| < 1$.

183. If $0 < |x - 3| < \delta$, then $|f(x) + 1| < 2$.

The following graph of the function f satisfies $\lim_{x \to 3} f(x) = 2$. In the following exercises, for each value of ε, find a value of $\delta > 0$ such that the precise definition of limit holds true.



184. $\varepsilon = 1.5$

185. $\varepsilon = 3$

[T] In the following exercises, use a graphing calculator to find a number δ such that the statements hold true.

186. $\left|\sin(2x) - \frac{1}{2}\right| < 0.1$, whenever $\left|x - \frac{\pi}{12}\right| < \delta$

187. $\left|\sqrt{x-4}-2\right| < 0.1,$ whenever $|x-8| < \delta$

In the following exercises, use the precise definition of limit to prove the given limits.

188. $\lim\limits_{x \to 2}(5x+8) = 18$

189. $\lim\limits_{x \to 3} \frac{x^2-9}{x-3} = 6$

190. $\lim\limits_{x \to 2} \frac{2x^2-3x-2}{x-2} = 5$

191. $\lim\limits_{x \to 0} x^4 = 0$

192. $\lim\limits_{x \to 2}(x^2+2x) = 8$

In the following exercises, use the precise definition of limit to prove the given one-sided limits.

193. $\lim\limits_{x \to 5^-} \sqrt{5-x} = 0$

194.
$\lim\limits_{x \to 0^+} f(x) = -2,$ where $f(x) = \begin{cases} 8x-3, & \text{if } x < 0 \\ 4x-2, & \text{if } x \geq 0 \end{cases}.$

195. $\lim\limits_{x \to 1^-} f(x) = 3,$ where $f(x) = \begin{cases} 5x-2, & \text{if } x < 1 \\ 7x-1, & \text{if } x \geq 1 \end{cases}.$

In the following exercises, use the precise definition of limit to prove the given infinite limits.

196. $\lim\limits_{x \to 0} \frac{1}{x^2} = \infty$

197. $\lim\limits_{x \to -1} \frac{3}{(x+1)^2} = \infty$

198. $\lim\limits_{x \to 2} -\frac{1}{(x-2)^2} = -\infty$

199. An engineer is using a machine to cut a flat square of Aerogel of area 144 cm^2. If there is a maximum error tolerance in the area of 8 cm^2, how accurately must the engineer cut on the side, assuming all sides have the same length? How do these numbers relate to $\delta,\ \ \varepsilon$, a, and L?

200. Use the precise definition of limit to prove that the following limit does not exist: $\lim\limits_{x \to 1} \frac{|x-1|}{x-1}.$

201. Using precise definitions of limits, prove that $\lim\limits_{x \to 0} f(x)$ does not exist, given that $f(x)$ is the ceiling function. (*Hint*: Try any $\delta < 1$.)

202. Using precise definitions of limits, prove that $\lim\limits_{x \to 0} f(x)$ does not exist: $f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}.$ (*Hint*: Think about how you can always choose a rational number $0 < r < d,$ but $|f(r)-0| = 1$.)

203. Using precise definitions of limits, determine $\lim\limits_{x \to 0} f(x)$ for $f(x) = \begin{cases} x \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}.$ (*Hint*: Break into two cases, x rational and x irrational.)

204. Using the function from the previous exercise, use the precise definition of limits to show that $\lim\limits_{x \to a} f(x)$ does not exist for $a \neq 0.$

For the following exercises, suppose that $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$ both exist. Use the precise definition of limits to prove the following limit laws:

205. $\lim\limits_{x \to a}(f(x)+g(x)) = L+M$

206. $\lim\limits_{x \to a}[cf(x)] = cL$ for any real constant c (*Hint*: Consider two cases: $c = 0$ and $c \neq 0$.)

207. $\lim\limits_{x \to a}[f(x)g(x)] = LM.$ (*Hint*: $|f(x)g(x)-LM| =$ $|f(x)g(x)-f(x)M+f(x)M-LM| \leq |f(x)||g(x)-M|+|M||f(x)-L|$.)

CHAPTER 2 REVIEW

KEY TERMS

average velocity the change in an object's position divided by the length of a time period; the average velocity of an object over a time interval $[t, a]$ (if $t < a$ or $[a, t]$ if $t > a)$, with a position given by $s(t)$, that is $v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}$

constant multiple law for limits the limit law $\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x) = cL$

continuity at a point A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied: (1) $f(a)$ is defined, (2) $\lim_{x \to a} f(x)$ exists, and (3) $\lim_{x \to a} f(x) = f(a)$

continuity from the left A function is continuous from the left at b if $\lim_{x \to b^-} f(x) = f(b)$

continuity from the right A function is continuous from the right at a if $\lim_{x \to a^+} f(x) = f(a)$

continuity over an interval a function that can be traced with a pencil without lifting the pencil; a function is continuous over an open interval if it is continuous at every point in the interval; a function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b), and it is continuous from the right at a and from the left at b

difference law for limits the limit law $\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$

differential calculus the field of calculus concerned with the study of derivatives and their applications

discontinuity at a point A function is discontinuous at a point or has a discontinuity at a point if it is not continuous at the point

epsilon-delta definition of the limit $\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$

infinite discontinuity An infinite discontinuity occurs at a point a if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$

infinite limit A function has an infinite limit at a point a if it either increases or decreases without bound as it approaches a

instantaneous velocity The instantaneous velocity of an object with a position function that is given by $s(t)$ is the value that the average velocities on intervals of the form $[t, a]$ and $[a, t]$ approach as the values of t move closer to a, provided such a value exists

integral calculus the study of integrals and their applications

Intermediate Value Theorem Let f be continuous over a closed bounded interval [a, b]; if z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$

intuitive definition of the limit If all values of the function $f(x)$ approach the real number L as the values of $x(\neq a)$ approach a, $f(x)$ approaches L

jump discontinuity A jump discontinuity occurs at a point a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

limit the process of letting x or t approach a in an expression; the limit of a function $f(x)$ as x approaches a is the value that $f(x)$ approaches as x approaches a

limit laws the individual properties of limits; for each of the individual laws, let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a; assume that L and M are real numbers so that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$; let c be a constant

multivariable calculus the study of the calculus of functions of two or more variables

one-sided limit A one-sided limit of a function is a limit taken from either the left or the right

power law for limits the limit law $\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n = L^n$ for every positive integer n

product law for limits the limit law $\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$

quotient law for limits the limit law $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ for $M \neq 0$

removable discontinuity A removable discontinuity occurs at a point a if $f(x)$ is discontinuous at a, but $\lim_{x \to a} f(x)$ exists

root law for limits the limit law $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even

secant A secant line to a function $f(x)$ at a is a line through the point $(a, f(a))$ and another point on the function; the slope of the secant line is given by $m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}$

squeeze theorem states that if $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$ over an open interval containing a and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$ where L is a real number, then $\lim_{x \to a} g(x) = L$

sum law for limits The limit law $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

tangent A tangent line to the graph of a function at a point $(a, f(a))$ is the line that secant lines through $(a, f(a))$ approach as they are taken through points on the function with x-values that approach a; the slope of the tangent line to a graph at a measures the rate of change of the function at a

triangle inequality If a and b are any real numbers, then $|a + b| \leq |a| + |b|$

vertical asymptote A function has a vertical asymptote at $x = a$ if the limit as x approaches a from the right or left is infinite

KEY EQUATIONS

- **Slope of a Secant Line**
  $m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}$
- **Average Velocity over Interval** $[a, t]$
  $v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}$
- **Intuitive Definition of the Limit**
  $\lim_{x \to a} f(x) = L$
- **Two Important Limits**
  $\lim_{x \to a} x = a \quad \lim_{x \to a} c = c$
- **One-Sided Limits**

$\lim_{x \to a^-} f(x) = L \quad \lim_{x \to a^+} f(x) = L$

- **Infinite Limits from the Left**
  $\lim_{x \to a^-} f(x) = +\infty \quad \lim_{x \to a^-} f(x) = -\infty$
- **Infinite Limits from the Right**
  $\lim_{x \to a^+} f(x) = +\infty \quad \lim_{x \to a^+} f(x) = -\infty$
- **Two-Sided Infinite Limits**
  $\lim_{x \to a} f(x) = +\infty : \lim_{x \to a^-} f(x) = +\infty$ and $\lim_{x \to a^+} f(x) = +\infty$

  $\lim_{x \to a} f(x) = -\infty : \lim_{x \to a^-} f(x) = -\infty$ and $\lim_{x \to a^+} f(x) = -\infty$
- **Basic Limit Results**
  $\lim_{x \to a} x = a \; \lim_{x \to a} c = c$
- **Important Limits**
  $\lim_{\theta \to 0} \sin\theta = 0$

  $\lim_{\theta \to 0} \cos\theta = 1$

  $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

  $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$

KEY CONCEPTS

2.1 A Preview of Calculus

- Differential calculus arose from trying to solve the problem of determining the slope of a line tangent to a curve at a point. The slope of the tangent line indicates the rate of change of the function, also called the *derivative*. Calculating a derivative requires finding a limit.
- Integral calculus arose from trying to solve the problem of finding the area of a region between the graph of a function and the x-axis. We can approximate the area by dividing it into thin rectangles and summing the areas of these rectangles. This summation leads to the value of a function called the *integral*. The integral is also calculated by finding a limit and, in fact, is related to the derivative of a function.
- Multivariable calculus enables us to solve problems in three-dimensional space, including determining motion in space and finding volumes of solids.

2.2 The Limit of a Function

- A table of values or graph may be used to estimate a limit.
- If the limit of a function at a point does not exist, it is still possible that the limits from the left and right at that point may exist.
- If the limits of a function from the left and right exist and are equal, then the limit of the function is that common value.
- We may use limits to describe infinite behavior of a function at a point.

2.3 The Limit Laws

- The limit laws allow us to evaluate limits of functions without having to go through step-by-step processes each time.
- For polynomials and rational functions, $\lim_{x \to a} f(x) = f(a)$.

- You can evaluate the limit of a function by factoring and canceling, by multiplying by a conjugate, or by simplifying a complex fraction.
- The squeeze theorem allows you to find the limit of a function if the function is always greater than one function and less than another function with limits that are known.

2.4 Continuity

- For a function to be continuous at a point, it must be defined at that point, its limit must exist at the point, and the value of the function at that point must equal the value of the limit at that point.
- Discontinuities may be classified as removable, jump, or infinite.
- A function is continuous over an open interval if it is continuous at every point in the interval. It is continuous over a closed interval if it is continuous at every point in its interior and is continuous at its endpoints.
- The composite function theorem states: If $f(x)$ is continuous at L and $\lim_{x \to a} g(x) = L$, then $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L)$.
- The Intermediate Value Theorem guarantees that if a function is continuous over a closed interval, then the function takes on every value between the values at its endpoints.

2.5 The Precise Definition of a Limit

- The intuitive notion of a limit may be converted into a rigorous mathematical definition known as the *epsilon-delta definition of the limit*.
- The epsilon-delta definition may be used to prove statements about limits.
- The epsilon-delta definition of a limit may be modified to define one-sided limits.

CHAPTER 2 REVIEW EXERCISES

True or False. In the following exercises, justify your answer with a proof or a counterexample.

208. A function has to be continuous at $x = a$ if the $\lim_{x \to a} f(x)$ exists.

209. You can use the quotient rule to evaluate $\lim_{x \to 0} \frac{\sin x}{x}$.

210. If there is a vertical asymptote at $x = a$ for the function $f(x)$, then *f* is undefined at the point $x = a$.

211. If $\lim_{x \to a} f(x)$ does not exist, then *f* is undefined at the point $x = a$.

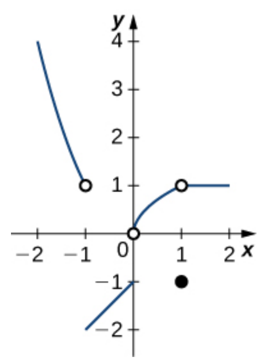
212. Using the graph, find each limit or explain why the limit does not exist.

a. $\lim_{x \to -1} f(x)$

b. $\lim_{x \to 1} f(x)$

c. $\lim_{x \to 0^+} f(x)$

d. $\lim_{x \to 2} f(x)$



In the following exercises, evaluate the limit algebraically or explain why the limit does not exist.

213. $\lim_{x \to 2} \frac{2x^2 - 3x - 2}{x - 2}$

214. $\lim_{x \to 0} 3x^2 - 2x + 4$

215. $\lim_{x \to 3} \frac{x^3 - 2x^2 - 1}{3x - 2}$

216. $\lim_{x \to \pi/2} \frac{\cot x}{\cos x}$

217. $\lim_{x \to -5} \frac{x^2 + 25}{x + 5}$

218. $\lim_{x \to 2} \frac{3x^2 - 2x - 8}{x^2 - 4}$

219. $\lim_{x \to 1} \frac{x^2 - 1}{x^3 - 1}$

220. $\lim_{x \to 1} \frac{x^2 - 1}{\sqrt{x} - 1}$

221. $\lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2}$

222. $\lim_{x \to 4} \frac{1}{\sqrt{x} - 2}$

In the following exercises, use the squeeze theorem to prove the limit.

223. $\lim_{x \to 0} x^2 \cos(2\pi x) = 0$

224. $\lim_{x \to 0} x^3 \sin\left(\frac{\pi}{x}\right) = 0$

225. Determine the domain such that the function $f(x) = \sqrt{x - 2} + xe^x$ is continuous over its domain.

In the following exercises, determine the value of c such that the function remains continuous. Draw your resulting function to ensure it is continuous.

226. $f(x) = \begin{cases} x^2 + 1, & x > c \\ 2x, & x \le c \end{cases}$

227. $f(x) = \begin{cases} \sqrt{x + 1}, & x > -1 \\ x^2 + c, & x \le -1 \end{cases}$

In the following exercises, use the precise definition of limit to prove the limit.

228. $\lim_{x \to 1} (8x + 16) = 24$

229. $\lim_{x \to 0} x^3 = 0$

230. A ball is thrown into the air and the vertical position is given by $x(t) = -4.9t^2 + 25t + 5$. Use the Intermediate Value Theorem to show that the ball must land on the ground sometime between 5 sec and 6 sec after the throw.

231. A particle moving along a line has a displacement according to the function $x(t) = t^2 - 2t + 4$, where x is measured in meters and t is measured in seconds. Find the average velocity over the time period $t = [0, 2]$.

232. From the previous exercises, estimate the instantaneous velocity at $t = 2$ by checking the average velocity within $t = 0.01$ sec.